ca foundation

quantitative aptitude

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“You’re much stronger than you think you are. Trust me.”
-Superman

00. BASICS OF QUANTITATIVE APTITUDE

CONCEPT 01 : NUMBER THEORY

TYPES OF NUMBERS

Type of Numbers		*Examples*
Positive Numbers	Numbers > 0	*1, 2, 3.14 etc.*
Negative Numbers	Numbers < 0	*-1, -2, -3.14 etc.*
Even Numbers	Numbers divisible by 2	*2, 4, 6 etc.*
Odd Numbers	Numbers not divisible by 2	*1, 3, 5 etc.*
Prime Numbers	Numbers divisible only by 1 & itself	*2, 3, 5, 7 etc.*
Composite Numbers	Numbers having > 2 factors	*4, 6, 8 etc.*
Rational Numbers	Numbers that can be expressed as fractions, where both Numerator & Denominator are Integers	*1, 2, 3, 4 etc.*
Irrational Numbers	Numbers other than Rational Numbers	*π*
Integers	All numbers (positive & negative) without fractions	*-2, -1, 0, 1, 2 etc.*
Whole Numbers	All positive numbers from 0 to infinity without fractions	*0, 1, 2, 3, 4 etc.*
Natural Numbers	All positive whole numbers from 1 till Infinity	*1, 2, 3, 4 etc.*

BODMAS

B	O	D	M	A	S
Brackets	**Orders (Exponents)**	**Division**	**Multiplication**	**Addition**	**Subtraction**

Author's Note

- ***Division and Multiplication have the same priority. If both appear in an expression, you perform them from left to right.***
- ***Addition and Multiplication have the same priority. If both appear in an expression, you perform them from left to right.***

{Master Question}

Q. Solve $10+4\times(6-2)\div2^2$ using BODMAS.

Sol. B - Brackets: $(6-2)=4$ The expression becomes: $10+4\times4\div2^2$

O - Orders: $2^2=4$ The expression becomes: $10+4\times4\div4$

D - Division / M - Multiplication (Left to Right):

= First, $4\times4=16$ (Multiplication comes first from left to right) The expression becomes: $10+16\div4$

= Next, $16\div4=4$ (Division) The expression becomes: $10+4$

A - Addition / S - Subtraction (Left to Right): Finally, $10+4=14$

DIVISIBILITY RULES

- **Divisibility by 2 :** If the last digit of a number is 0,2,4,6,8.
- **Divisibility by 3 :** If the sum of the digits is divisible by 3.
- **Divisibility by 4 :** If the last 2 digits of a number are 00 or divisible by 4.
- **Divisibility by 5 :** If the last digit of the number is either 0 or 5
- **Divisibility by 6 :** If the number is divisible by 2 and 3 both.
- **Divisibility by 7 :** If the difference between twice the last digit and the number formed by remaining digits is either 0 or divisible by 7.
- **Divisibility by 8 :** If the last 3 digits are 000 or divisible by 8.
- **Divisibility by 9 :** If the sum of the digits is divisible by 9.

CONCEPT 02 : ALGEBRA

PROPERTIES ON QUADRATIC EQUATION	PROPERTIES ON CUBIC EQUATION
▪ $(a + b)^2 = a^2 + 2ab + b^2$	▪ $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
▪ $(a - b)^2 = a^2 - 2ab + b^2$	▪ $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
▪ $a^2 - b^2 = (a + b)(a - b)$	▪ $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
▪ $(x + a)(x + b) = x^2 + (a + b)x + ab$	▪ $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

CONCEPT 03 : SIMPLE CALCULATOR TRICKS

Dirty Fractional Power ($a^{1/n}$)

- Press √ 12 times
- Subtract 1
- Divide by 'n'
- Add 1
- Press x= 12 times

Dirty Power (a^n)

- Press √ 12 times
- Subtract 1
- Multiply by 'n'
- Add 1
- Press x= 12 times

Common Log ($\log_{10} n$)

- Press √ 19 times
- Subtract 1
- Multiply by 227695

Antilog

- Divide by 227695
- Add 1
- Press x= 19 times

01. RATIO, PROPORTIONS, INDICES & LOGARITHMS

RATIO

CONCEPT 01 : BASICS OF RATIO

- A ratio is a comparison of the sizes of multiple quantities of the **same kind** by division.
- It is always expressed in the simplest / lowest form.
  E.g. 25:10 will be expressed as 5:2.

$a:b = \frac{a}{b}$

- a ----→ **Antecedent / First Term**
- b ----→ **Consequent / Second Term**

CONCEPT 02 : TYPES OF RATIO

	Type of Ratio	*Format*	*Examples*
1	**Compounded Ratio**	a:b & c:d compounded is ac:bd	*3:4, 5:6 & 3:11 compounded is* $\frac{3 \times 5 \times 3}{4 \times 6 \times 11}$ *= 15:88*
2	**Inverse Ratio**	b : a	*Inverse Ratio of 3:4 is 4:3*
3	**Duplicate Ratio**	$a^2 : b^2$	*Duplicate Ratio of 2:3 is* $2^2:3^2$ *= 4:9*
4	**Sub-Duplicate Ratio**	$\sqrt{a} : \sqrt{b}$	*Sub-Duplicate Ratio of 16:25 is* $\sqrt{16} : \sqrt{25}$ *= 4:5*
5	**Triplicate Ratio**	$a^3 : b^3$	*Triplicate Ratio of 3:4 is* $3^3:4^3$ *= 27:64*
6	**Sub-Triplicate Ratio**	$\sqrt[3]{a} : \sqrt[3]{b}$	*Sub-Triplicate Ratio of 125:64 is* $\sqrt[3]{125} : \sqrt[3]{64}$ *= 5:4*
7	**Continued Ratio**	a:b & b:c continued is a:b:c	▪ *3:4 and 4:5 continued is 3:4:5.* ▪ *3:4 and 5:6 continued is* $\frac{3 \times 5}{4 \times 5} : \frac{5 \times 4}{6 \times 4}$ *= 15:20 & 20:24, which is 15:20:24.*

PROPORTIONS

CONCEPT 03 : BASICS OF PROPORTIONS

- An equality of two ratios is called a Proportion.
  a : b = c : d also written as a : b :: c : d
- a, b, c & d are known as **terms**. a & d are called **Extremes** while b & c are called **Means**.
- d is known as **Fourth Proportional**.
- a, b, c & d must not be of the same kind in Proportions. a & b should be of the same kind and c & d should be of the same kind. **[Imp.]**
- **Continued Proportion :** If a : b = b : c, then
  (i) a is called **First Proportional**.
  (ii) c is called **Third Proportional**.
  (iii) b is called **Mean Proportional** = $\sqrt{ac}$ i.e. $b^2 = ac$
  (iv) a, b & c should be of the same kind.

CONCEPT 04 : PROPERTIES OF PROPORTIONS

If a : b = c : d

Property 01 {Cross Product Rule}	Property 02 {Invertendo}	Property 03 {Alternendo}
ad = bc *{Product of Extremes = Product of Means}*	b : a = d : c	a : c = b : d
Property 04 {Componendo}	***Property 05 {Dividendo}***	***Property 06 {Componendo & Dividendo}***
(a + b) : b = (c + d) : d	(a – b) : b = (c – d) : d	(a + b) : (a – b) = (c + d) : (c – d)

If a : b = c : d = e : f = ...

Property 07 {Addendo}	Property 08 {Subtrahendo}
(a + c + e + ...) : (b + d + f + ...)	(a - c - e - ...) : (b - d - f - ...)

INDICES

CONCEPT 05 : EXPONENTIAL FORM

$a^b = c$ → Solution

Power / Index: b

Base: a

- *Example : $2^3 = 8$, where 2 is base, 3 is power / index and 8 is the solution.*
- $\sqrt[r]{a} = a^{1/r}$

  Example : $\sqrt[3]{8} = 8^{1/3} = 2$

CONCEPT 06 : LAWS OF INDICES

Law		Example
1	$a^m \times a^n = a^{m+n}$	$3^5 \times 3^6 = 3^{5+6} = 3^{11}$
2	$a^m / a^n = a^{m-n}$	$3^5 / 3^3 = 3^{5-3} = 3^2$
3	$(a^m)^n = a^{mn}$	$(3^5)^3 = 3^{5\times3} = 3^{15}$
4	$(ab)^n = a^n \times b^n$	$6^3 = (2 \times 3)^3 = 2^3 \times 3^3$
5	$a^0 = 1$	$5^0 = 1$
6	$a^{-m} = 1/a^m$	$3^{-2} = 1/3^2$ or $3^2 = 1/3^{-2}$
7	If $a^m = b^m$, then $a=b$ $(a,b \neq -1, 0, 1)$	$a^2 = 3^2$, then $a = 3$
8	If $a^x = a^y$, then $x=y$ $(x, y \neq 0, 1)$	$a^3 = a^x$, then $x = 3$

Author's Note : For simplifying the problems of Indices, students are advised to learn the following values by ♥

Indices of 2		Indices of 3	
$2^1 = 2$	$2^6 = 64$	$3^1 = 3$	$3^6 = 729$
$2^2 = 4$	$2^7 = 128$	$3^2 = 9$	$3^7 = 2187$
$2^3 = 8$	$2^8 = 256$	$3^3 = 27$	
$2^4 = 16$	$2^9 = 512$	$3^4 = 81$	
$2^5 = 32$	$2^{10} = 1024$	$3^5 = 243$	

LOGARITHMS

CONCEPT 07 : LOGARITHMIC FORM

How to read Logarithm?

$\log_b a = c$

is read as, ***"log a base b equals c"***

$\log_4 16 = 2$

will be read as log 16 base 4 equals 2

Conversion of Exponential Form into Logarithmic Form

Index of Exponential Form becomes Solution of Logarithmic Form

$a^b = c$ → $\log_a c = b$

Base of Exponential Form becomes Base of Logarithmic Form

$4^2 = 16$ will be converted as $\log_4 16 = 2$

CONCEPT 08 : TYPES OF LOGARITHMS

TYPE 01 : COMMON LOGARITHM

In Common Log base is always taken as 10.

E.g. $\log_{10} 16$

TYPE 02 : NATURAL LOGARITHM

In Natural Log base is always taken as 'e'.

e = Exponential No. = 2.33 (approx.)

E.g. $\log_e 16$

CONCEPT 09 : LAWS OF LOGARITHMS

Law		Examples
1	$\log_a mn = \log_a m + \log_a n$	$\log 15 = \log (3 \times 5) = \log 3 + \log 5$ **{ $\log 3 \times \log 5 \neq \log 3 + \log 5$ }**
2	$\log_a {}^m/_n = \log_a m - \log_a n$	$\log {}^3/_5 = \log 3 - \log 5$ **{ $\log 3 / \log 5 \neq \log 3 - \log 5$ }**
3	$\log_a m^n = n \times \log_a m$	$\log 9 = \log 3^2 = 2 \times \log 3$
4	$\log_a a = 1$	$\log_9 9 = 1$
5	$\log_a 1 = 0$	$\log_9 1 = 0$
6	$\log_b a = \frac{\log a}{\log b} = \frac{1}{\log_a b}$ {Base Changing}	$\log_2 3 = \frac{\log 3}{\log 2} = \frac{1}{\log_3 2}$
7	$\log_b a \times \log_a b = 1$	$\log_2 3 \times \log_3 2 = 1$
8	$a^{\log_a m} = m$ {Inverse Log}	(i) $3^{\log_3 2} = 2$ (ii) $3^{2\log_3 x} = 3^{\log_3 x^2} = x^2$

Note : If Base of Exponent & Base of Logarithm are same, then only this property is applicable.

02. EQUATIONS

CONCEPT 01 : BASICS OF EQUATIONS

Expression	It is a combination of numbers, variables & operations that represents a value. *E.g. 5x - 3*
Equation	It is a mathematical statement of equality. ▪ **Conditional Equation :** If the equality is true for a certain value of variable. *E.g. x + 1 = 2 holds true for only x = 1. So it is a Conditional Equation.* ▪ **Identity :** If the equality is true for all the values of the variable involved. *E.g.* $\frac{x+2}{3} + \frac{x+3}{2} = \frac{5x+13}{6}$ *holds true for all values of x. So it is an Identity.*
Inequality	It is a statement that compares two values or expressions, indicating they are not equal. *E.g.* $2x + 3 > 7$, $2x + 3 < 10$, $2x + 3 \neq 8$

TYPES OF EQUATIONS

	Degree (Highest Power of Variable)	*Roots (Solutions)*	
Linear Equation	1	1	*8x + 17(x-3) = 4(4x-9) + 12*
Quadratic Equation	2	2	$3x^2 + 5x + 6 = 0$
Cubic Equation	3	3	$4x^3 + 3x^2 + x - 7 = 1$

Note : *Two or more Linear Equations involving multiple variables are called* ***Simultaneous Linear Equations.***
E.g. x + 2y =1, 2x + 3y = 2 are jointly called Simultaneous Linear Equations.

CONCEPT 02 : SIMPLE EQUATIONS & SIMULTANEOUS LINEAR EQUATIONS

- A simple equation is of the general form ax + b = 0, where a ≠ 0.
- It has only one root.

Author's Note : Questions from this segment can be solved using Option Approach i.e. Check the values of the variable by putting options in the equation. So, learner should not invest time in learning the subjective methods like Elimination Method, Cross Multiplication Method etc.

CONCEPT 03 : QUADRATIC EQUATION

- A quadratic equation is of the general form $ax^2 + bx + c = 0$.
  - → When b = 0, it is known as **Pure Quadratic Equation**.
  - → When b ≠ 0, it is known as **Affected Quadratic Equation**.
- It has 2 roots.

QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

E.g. Find the roots of the equation $x^2 - 5x + 6 = 0$.

Sol. Here, a = 1, b = -5 & c = 6. Putting the values in Quadratic Formula, we get,

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 1 \times 6}}{2 \times 1}$$

= On solving we get,

= 3, 2

Hence, the two roots are x = 3 & x = 2.

OTHER POINTS

(i) Sum of Roots of a Quadratic Equation $(\alpha + \beta) = \frac{-b}{a}$

(ii) Product of Roots of a Quadratic Equation $(\alpha \times \beta) = \frac{c}{a}$

(iii) **Construction of a Quadratic Equation when Sum & Products are given**

$$x^2 - (\text{Sum of Roots})x + (\text{Product of Roots}) = 0$$

NATURE OF ROOTS

Discriminant (D) : Since $b^2 - 4ac$ discriminates between the roots of the quadratic equation, it is called Discriminant.

Case		*Nature of Roots*
A	D $(b^2 - 4ac) = 0$	Real & Equal
B	D $(b^2 - 4ac) < 0$	Imaginary
C	D $(b^2 - 4ac) > 0$	
	▪ It is a Perfect Square.	Real, Rational & Unequal (Distinct)
	▪ It is not a Perfect Square.	Real, Irrational & Unequal (Distinct)

OTHER POINTS

(i) If one root is reciprocal of another, then c = a.

(ii) If one root is equal to another but of opposite sign, then b = 0.

(iii) Irrational Roots occur in conjugate pairs i.e. if $(m + \sqrt{n})$ is one root, then $(m - \sqrt{n})$ will be the other root.

CONCEPT 04 : CUBIC EQUATION

- A cubic equation is of the general form $ax^3 + bx^2 + cx + d = 0$.
- It has 3 roots.

Author's Note : Questions from this segment have to be solved using Option Approach i.e. Check the values of the variable by putting options in the equation.

03. LINEAR INEQUALITIES

CONCEPT 01 : BASICS OF LINEAR INEQUALITIES

Inequality	Statements where two quantities unequal are but a relationship exists between them.
Linear Inequality	Any Linear Function (Degree = 1) which involves an inequality sign (≤, <, >, ≥, ≠) is known as a Linear Inequality.
Solution Space	The values of the variables that satisfy an inequality are called the Solution Space (S.S).

Brackets

[a, b]	Both a & b are included.	**[a, b)**	a is included but b is excluded.
(a, b)	Both a & b are excluded.	**(a, b]**	a is excluded but b is included.

OTHER POINTS

(a) Infinities are always written with an open bracket i.e. *(-∞, ∞)*.

(b) Closed Brackets are represented on the Number Line with ●, while Open Brackets are represented with ○.

(c) Within the Brackets, a < b.

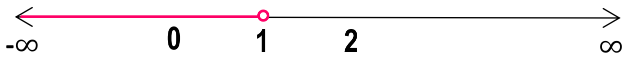
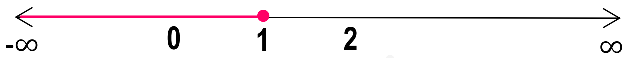
Author's Note : If the inequality contains ≤, ≥, use Closed Brackets and when it contains <, >, use Open Brackets.

CONCEPT 02 : LINEAR INEQUALITIES IN ONE VARIABLE

General Form

ax + b ≤ 0	ax + b ≥ 0	Where, 'a' is a non-zero real number.
ax + b < 0	ax + b > 0	

Representation on Number Line

x < 1	**(-∞, 1)**	
x ≤ 1	**(-∞, 1]**	
x > 1	**(1, ∞)**	
x ≥ 1	**[1, ∞)**	

Rules for Solving

If same no. on both sides is	+ve Number	-ve Number
Added / Subtracted	Doesn't affect the sign	Doesn't affect the sign
Multiplied / Divided	Doesn't affect the sign	Reverse the sign

Author's Note : Learners may consider using Optional Approach like Equations.

CONCEPT 03 : LINEAR INEQUALITIES IN TWO VARIABLES

General Form

ax + by ≤ c	ax + by ≥ c	Where, 'a' & 'b' are non-zero real numbers.
ax + by < c	ax + by > c	

Representation on Graph

Let us consider an inequality 3x + y < 6.

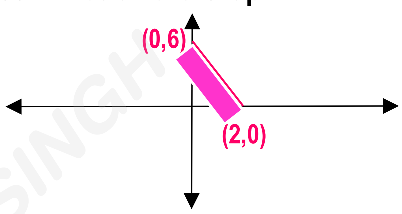
Step 01 : Convert it into an Equation

→ 3x + y = 6

Step 02 :

x	0	2
y	6	0

Step 03 : Plot on the Graph



Step 04 : Put (0,0) in the inequality, if it satisfies, then shade towards the origin, otherwise away. If the line passes through origin, then take any point from x axis or y axis.

Author's Note :

	c is +ve	*c is -ve*
ax + by < c	*Shading Towards Origin*	*Shading Away from Origin*
ax + by > c	*Shading Away from Origin*	*Shading Towards Origin*

04. MATHEMATICS OF FINANCE

SIMPLE INTEREST (SI)

Meaning

- It is the interest computed on the principal for the entire period of borrowing.
- It is calculated on the outstanding principal balance & not on interest previously earned.

Computation

$A = P + SI$ ········ $SI = \frac{P \cdot r \cdot t}{100}$

$A = P + \frac{P \cdot r \cdot t}{100}$

$A = P\left(1 + \frac{r \cdot t}{100}\right)$

KEY

A	Accumulated Amount (Final Value)
P	Principal (Initial Value)
r	Annual Rate of Interest (aka 'i')
T	Time (in years)

OTHER POINTS

(a) Simple Interest is calculated on Principal (P) irrespective of the Time.
SI (Year 1) = SI (Year 2) = ... = SI (Year n)

(b) If the question provides both A & P, then,

Alternative 1 : Directly apply the formula $A = P\left(1 + \frac{r \cdot t}{100}\right)$

Alternative 2 : Calculate SI from SI = A – P & then apply the formula of SI.

E.g. Sania deposited 50,000 in a bank for two years with the interest rate of 5.5% p.a. Calculate,
(a) The interest she would earn.
(b) The final value of investment.

Sol. (a) Interest earned

$\Rightarrow SI = P \cdot r \cdot t$

$\Rightarrow SI = ₹50{,}000 \times \frac{5.5}{100} \times 2 = \mathbf{₹5{,}500}$

(b) Final Value of Investment

$\Rightarrow A = P + SI$

$\Rightarrow A = ₹50{,}000 + ₹5{,}500 = \mathbf{₹55{,}500}$

E.g. If a certain sum becomes ₹575 at the rate of 5% p.a. in the same time when ₹750 becomes ₹840 at the rate of 4% p.a. Calculate the sum.

Sol. Calculation of Time (t)

ALTERNATIVE I

$\Rightarrow A = P\left(1 + \frac{r \cdot t}{100}\right)$

$\Rightarrow 840 = 750\left(1 + \frac{4 \cdot t}{100}\right)$

On Solving,

$\Rightarrow$ **t = 3 years**

ALTERNATIVE II

$\Rightarrow SI = ₹840 - ₹750 = ₹90$

$\Rightarrow \frac{P \cdot r \cdot t}{100} = ₹90$

$\Rightarrow \frac{750 \cdot 4 \cdot t}{100} = ₹90$

On Solving,

$\Rightarrow$ **t = 3 years**

Calculation of Principal (P)

$\Rightarrow A = P\left(1 + \frac{r \cdot t}{100}\right)$

$\Rightarrow 575 = P\left(1 + \frac{4 \cdot 3}{100}\right)$

On Solving,

$\Rightarrow P = ₹500$

COMPOUND INTEREST (CI)

Meaning

Compound Interest is the interest calculated on the Principal as well as on the Accumulated Interest from previous periods.

Computation

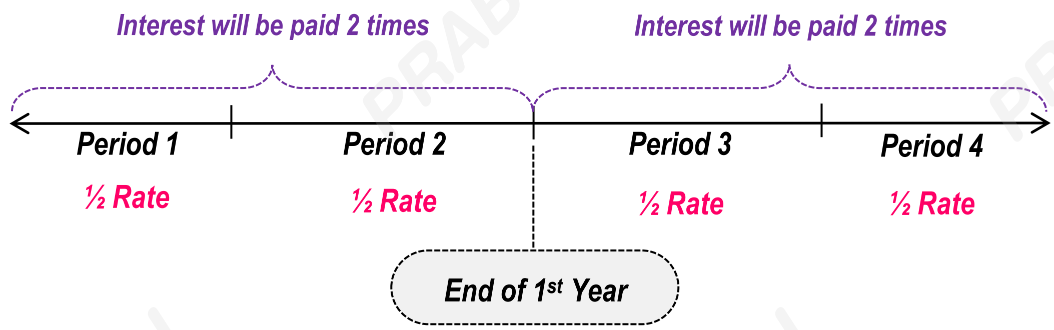
$$A = P(1 + r)^n$$
$$CI = A - P$$
$$= P(1 + r)^n - P$$
$$= P[(1 + r)^n - 1]$$

Period of Compounding	Conversions
Semi-Annually / Half Yearly	2
Quarterly	4
Monthly	12
Daily / Continuously	365

KEY

- **A** Accumulated Amount (Final Value)
- **P** Principal (Initial Value)
- **r** Rate of Interest per Conversion $= \dfrac{i}{\text{No. of Conversions}}$
- **n** Total No. of Conversions = No. of Years x No. of Conversions

E.g. Mr. X invested ₹1,00,000 @10% p.a. compounded semi-annually for 2 years.



Sol. <u>Computation of 'r'</u>

$\Rightarrow \dfrac{i}{\text{No. of Conversions}}$

$\Rightarrow \dfrac{10\%}{2}$ **= 5% p.a.**

<u>Computation of 'n'</u>

$\Rightarrow$ No. of Years x No. of Conversions

$\Rightarrow$ 2 x 2 = **4**

IN CASE OF DEPRECIATION (WDV)

$$A = P(1 - r)^n$$

[Pro Tip] 'n' usually comes in decimal form. Refer Chapter 0 to learn how to solve.

KEY

- **A** Residual Value
- **P** Cost
- **r** Rate of Depreciation (WDV)
- **n** No. of Years

CONTINUOUS COMPOUNDING

$$A = P \cdot e^{rt}$$
$$(e = 2.7183)$$

Author's Note : ICAI generally provides the value of e^{rt} in question itself.

EFFECTIVE RATE OF INTEREST

Meaning

If interest is compounded more than once in a year, then the equivalent rate of interest compounded annually is known as Effective Rate of Interest.

Computation

$$E = (1 + r)^n - 1$$

E.g. Calculate Effective Rate of Interest if the amount is compounded semi-annually @6% p.a.

Sol. Here, $r = \dfrac{i}{\text{No. of Conversions}} = \dfrac{6\%}{2} = 0.03$ & n = 1 Year x 2 Conversions = 2

Now,

$\Rightarrow$ **$E = (1 + r)^n - 1$**

$\Rightarrow$ $= (1 + 0.03)^2 - 1 =$ **0.0609 or 6.09%**

ANNUITY

Meaning & Types

Constant ₹₹ Paid / Received over the period + **Equal Intervals between consecutive Payments / Receipts**

ANNUITY

REGULAR / ORDINARY ANNUITY

When the First Payment is Paid / Received **at the end of the period**.

E.g. EMI, Sinking Fund etc.

ANNUITY DUE / IMMEDIATE ANNUITY

When the First Payment is Paid / Received **at the beginning of the period**.

E.g. RD, Insurance Premium etc.

Future Value and Present Value

Let us consider the formula studied earlier,

$$A = P(1 + r)^n$$

A is the Final Amount to be received after n conversions, hence A is known as Future Value (FV).

P is the Initial Amount Invested as of today, hence, it is known as Present Value (PV).

$\Rightarrow$

$$FV = PV(1 + r)^n$$

$$PV = \frac{FV}{(1 + r)^n}$$

E.g. What is the Present Value of ₹1 to be received after 2 years compounded annually at 10% interest rate?

Sol. Here, FV = ₹1, r = 10%, n = 2

$$PV = \frac{FV}{(1 + r)^n}$$

$$\Rightarrow \quad = \frac{1}{(1 + 0.1)^2} = ₹0.8264 = \textbf{₹0.83 (approx.)}$$

Standard Formulae

	FUTURE VALUE (FV)	PRESENT VALUE (PV)
ORDINARY / REGULAR ANNUITY	$FV = \frac{A[(1 + r)^n - 1]}{r}$	$PV = \frac{A[1 - (1 + r)^{-n}]}{r}$
ANNUITY DUE / IMMEDIATE ANNUITY	$FV = \frac{A[(1 + i)^n - 1]}{r} \cdot (1 + i)$	$PV = \frac{A[1 - (1 + r)^{-(n-1)}]}{r} + A$

MASTER SUMMARY

A	**Investment Made**	▪ **General Case :** Ordinary Annuity + Future Value *(Investments are assumed to be made at the end of period)* ▪ **Recurring Deposit :** Immediate Annuity + Future Value *(ICAI uses the words 'starting from today')*
B	**Loan Taken**	▪ **General Case :** Ordinary Annuity + Present Value *(EMI is assumed to be paid at the end of the period)*

Sinking Fund

It is a fund created for :
(a) Acquisition of any asset at the end of its useful life (Replacement of Asset);
(b) Payment of any Liability in future

Ordinary Annuity + Future Value

Net Present Value (NPV)

COMPUTATION OF NPV

PV of Net Cash Inflow	xxx
(-) PV of Net Cash Outflow	(xxx)
Net Present Value	**xxx**

DECISION RULE

NPV > 0	Accept the Proposal
NPV = 0	Indifferent
NPV < 0	Reject the Proposal

E.g. Compute the NPV for a project with Net Investment of ₹1,00,000 and Net Cash Flows for Year 1 is ₹55,000, Year 2 is ₹80,000 and for Year 3 is ₹15,000. Further, the company's cost of capital is 10%. Should the company accept the project?

Sol. Computation of NPV

Year	Net Cash Flows (₹) (1)	PVIF @10% (2)	Discounted Cash Flows (₹) (3)
0	(1,00,000)	1.000	(1,00,000)
1	55,000	0.909	49,995
2	80,000	0.826	66,080
3	15,000	0.751	11,265
		Net Present Value (NPV)	**27,340**

Recommendation : Since, the NPV is Positive. The company should accept the Project.

Nominal & Real Rate

NOMINAL RATE OF INTEREST

It is the Stated Rate of Interest on a given loan, bond etc. It works as per Simple Interest.

REAL RATE OF INTEREST

It is the interest rate the lender or investor gets after factoring in Inflation.

Nominal Interest Rate = Real Interest Rate + Inflation

Bond

- A bond is a debt security in which the issuer owes & is obliged to repay the principal & interest.
- They are generally issued for a fixed term (generally longer than a year).
- **Present Value of a Bond**

$$= \frac{\text{Interest}}{(1+r)^1} + \frac{\text{Interest}}{(1+r)^2} + \ldots + \frac{\text{Interest}}{(1+r)^n} + \frac{\text{Principal}}{(1+r)^n}$$

Perpetuity

- It is an annuity in which the periodic payments / receipts begin on a fixed date & continue indefinitely.
- **Present Value of Perpetuity**

$$= \frac{\text{A (Annuity)}}{r}$$

CAGR

- Compounded Annual Growth Rate (CAGR) calculates the mean annual growth rate of an investment over a specified time period.
- **Computation of CAGR**

$$= \left(\frac{\text{Future/End Value}}{\text{Present/Beginning Value}}\right)^{\frac{1}{n}} - 1$$

05. PERMUTATIONS & COMBINATIONS

CONCEPT 01 : FACTORIAL

Definition	▪ The factorial 'n' represents the product of all integers from 1 to n. ▪ It is denoted as n! or ⌊n. ▪ $n! = n \cdot (n-1) \cdot (n-2) \ldots\ldots \cdot 3 \cdot 2 \cdot 1$ ▪ $0! = 1$	***Author's Note : Students are advised to learn values from 1! to 7!.***

CONCEPT 02 : FUNDAMENTAL PRINCIPLES OF COUNTING

	ADDITION RULE (OR)	MULTIPLICATION RULE (AND)
Meaning	If there are two alternative jobs which can be done in **'m'** ways & in **'n'** ways respectively, then either of the two jobs can be done in **'m + n'** ways.	If a certain thing can be done in **'m'** ways, & when it has been done, a second thing can be done in **'n'** ways, then total ways of doing both things = **'mn'**
Example	*If one wants to go to brief academy by bus where there are 5 buses **or** by auto where there are 4 autos, then total ways = 5 + 4 = **9***	*If one wants to go to brief academy by bus where there are 5 buses **and** come back by auto where there are 4 autos, then total ways = 5*4 = **20***

CONCEPT 03 : PERMUTATIONS

Meaning	▪ Each of the arrangements which can be made by taking some or all of a number of things. ▪ It is denoted by ^nP_r or $_nP_r$ or $P_{(n,r)}$. ▪ ^nP_r is read as 'r' objects arranged out of 'n' different objects. ▪ $^nP_r = \frac{n!}{(n-r)!}$, $0 \le r \le n$ where, n,r ∈ Integers **(Z)**. ▪ $^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1)$ **[No. of Factors = 'r']** ▪ $^nP_r = {}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$

MASTER SUMMARY

Type 01	**<u>No. of Permutations when all the objects are distinct</u>** (a) 'n' different objects taken 'r' at a time. (b) 'n' different objects taken all at a time. (c) 'n' different objects taken 'r' at a time but repetition is allowed.	^nP_r ^nP_n **(n!)** n^r
Type 02	**<u>No. of Permutations when all the objects are not distinct</u>** *(p_1 objects are of one kind, p_2 objects are of second kind,……, p_k objects are of k^{th} kind)*	$\frac{n!}{p_1 \cdot p_2 \cdot \ldots \cdot p_k}$
Type 03	**<u>Permutations with Restrictions</u>** (a) 'n' different objects taken 'r' at a time, when a particular object is not included in any arrangement. (b) 'n' different objects taken 'r' at a time, when a particular object is always included in all arrangement.	$^{n-1}P_r$ $^{n-1}P_{r-1}$
Type 04	**<u>Circular Permutations</u>** (a) No. of Circular Permutations of 'n' different objects taken all at a time. (b) No. of ways of arranging 'n' persons along a round table so that no person has the same two neighbours. (c) No. of necklaces formed with 'n' different beads.	$(n-1)!$ $\frac{(n-1)!}{2}$ $\frac{(n-1)!}{2}$

CONCEPT 04 : COMBINATIONS

Meaning	▪ Each of the different selections made by taking some or all of a no. of objects, irrespective of their arrangements. ▪ It is denoted as It is denoted by ^nC_r or C (n,r), $C_{n,r}$. ▪ ^nC_r is read as 'r' objects selected out of 'n' different objects. ▪ $^nC_r = \frac{n!}{(n-r)!\,r!}$, $0 \le r \le n$ where, n,r ∈ Integers **(Z)**.

Standard Results	▪ $^nP_r = {^nC_r} \cdot r!$ ▪ $^nC_r = {^nC_{n-r}}$ ▪ $^nC_n = {^nC_0} = 1$ ▪ If $^nC_a = {^nC_b}$, then either a = b or n = a+b ▪ $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

MASTER SUMMARY

Type 01	Total ways to form groups by taking some or all of 'n' things	$2^n - 1$
Type 02	Total ways to make groups by takin some or all of 'n', where *(n_1 objects are of one kind, n_2 objects are of second kind,……)*	$\{(n_1+1)(n_2+1)\ldots\}-1$
Type 03	**Geometry based Problems** (a) No. of Straight Lines with given 'n' points. (b) No. of Straight Lines with given 'n' points where 'm' points are collinear. (c) No. of Triangles with given 'n' points. (d) No. of Triangles with given 'n' points where 'm' points are collinear. (e) No. of Parallelograms with the given set of 'm' Parallel Lines & another set of 'n' Parallel Lines (f) No. of Diagonals with 'n' sides (g) Maximum No. of points of intersection of 'n' circles	 $^nC_2$ $^nC_2 - {^mC_2} + 1$ $^nC_3$ $^nC_3 - {^mC_3}$ $^nC_2 \cdot {^mC_2}$ $^nC_2 - n$ $n \cdot (n-1)$
Type 04	No. of ways of dividing 'n' different items into 'k' groups of 'h' items each	$\frac{n!}{k! \cdot (h!)^k}$
Type 05	Total No. of handshakes between 'n' different persons	$\frac{n \cdot (n-1)}{2}$

Author's Note : If the question asks the total no. of factors of any number, let's say, 75,600. Then firstly write it in its prime factorisation form = $2^4 \cdot 3^3 \cdot 5^2 \cdot 7^1$. Now, No. of Factors will be given by multiplying exponents after increasing them by 1. In this case, 5 x 4 x 3 x 2 = 120.

Author's Note : For our learners reference, basic knowledge about a deck of cards is as follows :

Total Cards — ***52 (No Jokers in a Standard Deck)***
- ***Ace Cards (A) : 1 in each suit & 4 in deck***
- ***Face Cards (K, Q, J) : 3 in each suit & 12 in deck***
- ***Number Cards (2-10) : 9 in each suit & 36 in deck***

Colours — ***Red Cards (26) + Black Cards (26)***

Suits — ***Red Suits [Hearts (13) + Diamonds (13)] + Black Suits [Spades (13) + Clubs(13)]***

06. SEQUENCE & SERIES

CONCEPT 01 : BASICS OF SEQUENCE & SERIES

	SEQUENCE	SERIES
Meaning	An ordered collection of numbers if arranged in some definite rule or law.	The sum of elements of the Sequence is known as Series. It is denoted by the Greek Letter $\sum$.
Examples	▪ *2, 4, 6, 8, 10 {Finite Sequence}* ▪ *$1^2, 2^2, 3^2, 4^2, \ldots\ldots$ {Infinite Sequence}*	▪ *2 + 4 + 6 + 8 + 10 {Finite Series}* ▪ *1 – 2 + 3 – 4 ……. {Infinite Series}*
	Author's Note : At CA Foundation Level only two Sequences are discussed i.e. AP & GP.	

CONCEPT 02 : ARITHMETIC PROGRESSION (A.P.) & GEOMETRIC PROGRESSION (G.P.)

	ARITHMETIC PROGRESSION (AP)	GEOMETRIC PROGRESSION (GP)
Definition	Any sequence is said to be an AP if the **difference** between the consecutive terms is same.	Any sequence is said to be a GP if the **ratio** between the consecutive terms is same.
General Term	$a_n = a + (n-1)d$	$a_n = ar^{n-1}$
General Form	$a, a+d, a+2d, a+3d, \ldots\ldots, a+(n-1)d$	$a, ar, ar^2, ar^3, \ldots\ldots, ar^{n-1}$
n^{th} Term from the end	$l - (n-1)d$	$l\left(\frac{1}{r}\right)^{n-1}$
Sum of 'n' terms	$S_n = \frac{n}{2}[2a + (n-1)d]$ ***Note :*** *If 1^{st} & last term are given,* $S_n = \frac{n}{2}[a + l]$	**If r > 1:** $S_n = \frac{a[r^n - 1]}{r - 1}$ **If r < 1:** $S_n = \frac{a[1 - r^n]}{1 - r}$
	Author's Note : While calculating S_n, if 'n' is not given in question, then first calculate 'n' using General Term as discussed above.	
n^{th} Term if S_n is given	$a_n = S_n - S_{n-1}$	-
Sum of ∞ Terms	-	$S_\infty = \frac{a}{1 - r}$
Key	▪ a = First Term ▪ n = No. of Terms ▪ d = Common Difference $= a_2 - a_1 = a_3 - a_2 = a_n - a_{n-1}$ ▪ a_n = n^{th} Term / Last Term (l) / t_n ▪ S_n = Sum of 'n' Terms	▪ a = First Term ▪ n = No. of Terms ▪ d = Common Ratio $= \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_n}{a_{n-1}}$ ▪ a_n = n^{th} Term / Last Term (l) / t_n ▪ S_n = Sum of 'n' Terms

CONCEPT 03 : SUM OF SOME IMPORTANT SERIES

	Sum of 'n' Consecutive Nos.	Sum of Squares	Sum of Cubes
Natural Nos.	$\frac{n(n+1)}{2}$ $\{1 + 2 + 3 + \ldots\ldots + n\}$	$\frac{n(n+1)(2n+1)}{6}$ $\{1^2 + 2^2 + 3^2 + \ldots\ldots + n^2\}$	$\left[\frac{n(n+1)}{2}\right]^2$ $\{1^3 + 2^3 + 3^3 + \ldots\ldots + n^3\}$
Even Nos.	$n(n+1)$ $\{2 + 4 + 6 + \ldots\ldots + n\}$	$\frac{2n(n+1)(2n+1)}{3}$ $\{2^2 + 4^2 + 6^2 + \ldots\ldots + n^2\}$	$2[n(n+1)]^2$ $\{2^3 + 4^3 + 6^3 + \ldots\ldots + n^3\}$
Odd Nos.	n^2 $\{1 + 3 + 5 + \ldots\ldots + n\}$	$\frac{n(2n+1)(2n-1)}{3}$ $\{1^2 + 3^2 + 5^2 + \ldots\ldots + n^2\}$	$n^2(2n^2 - 1)$ $\{1^3 + 3^3 + 5^3 + \ldots\ldots + n^3\}$

CONCEPT 04 : ARITHMETIC MEAN (A.M.) & GEOMETRIC MEAN (G.M.)

	ARITHMETIC MEAN (A.M.)	GEOMETRIC MEAN (G.M.)
	Author's Note : For this topic, 'Mean' implies terms other than extremes & not average.	
Single Mean between two terms (a, b)	a, x, b $x = \frac{a+b}{2}$, where 'x' is mean	a, x, b $x = \sqrt{ab}$, where 'x' is mean
'n' Mean between two terms (a, b)	a, ……. , b ▪ **Step 01 :** Calculate 'd' [Using $d = \frac{b-a}{n+1}$] ▪ **Step 02 :** Now, ⇒ 1^{st} A.M. = a + d [2^{nd} term of AP] ⇒ 2^{nd} A.M. = a + 2d [3^{rd} term of AP] ⋮ ⇒ n^{th} A.M. = a + nd [$(n-1)^{th}$ term of AP]	a, ……. , b ▪ **Step 01 :** Calculate 'r' [Using $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$] ▪ **Step 02 :** Now, ⇒ 1^{st} A.M. = ar [2^{nd} term of GP] ⇒ 2^{nd} A.M. = ar^2 [3^{rd} term of GP] ⋮ ⇒ n^{th} A.M. = ar^n [$(n-1)^{th}$ term of GP]

SOME IMPORTANT RESULTS TO REMEMBER

Type 01	If the m^{th} term of an AP is 'n' & n^{th} term is 'm', then r^{th} term of it is	m + n - r
Type 02	If the m^{th} term of an AP is 'n' & n^{th} term is 'm', then $(m + n)^{th}$ term is	0
Type 03	x + xx + xxx + …… to n terms , then S_n is given by	$\frac{x}{n}\left[\left\{\frac{10}{9}\left(10^n-1\right)\right\}-n\right]$
Type 04	0.x + 0.xx + 0.xxx + ……. to n terms, then S_n is given by	$\frac{x}{n}\left[\left\{n-\left(\frac{1-0.1^n}{9}\right)\right\}\right]$
Type 05	To Evaluate questions of the format : ▪ $0.a\overline{b} = \frac{ab-a}{90}$ ▪ $0.ab\overline{c} = \frac{abc-ab}{900}$ ▪ $0.ab\overline{cd} = \frac{abcd-ab}{9900}$	***Author's Note : In denominator, 9 will be written as many times as digits under the bar & 0 will appear as many times as digits outside the bar.*** *E.g.* $0.21\overline{75} = \frac{2175-21}{9900}$ *E.g.* $0.38\overline{7} = \frac{387-38}{900}$
Type 06	If A be the AM & G be the GM between two terms, then A > G.	
Type 07	If a, b, c are in AP as well as in GP, then (a) they are also in Harmonic Progression (HP); and (b) their reciprocals are also in AP.	

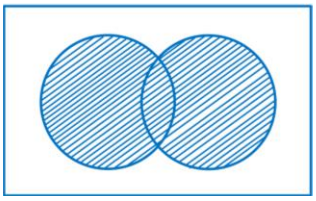
07A. SETS, RELATIONS & FUNCTIONS

SETS

CONCEPT 01 : BASICS OF SETS	
Meaning	Collection of well-defined **distinct** objects (finite/ infinite). Each object is called an **element** of the set. **OTHER POINTS** 1. The elements of the two sets may be listed in any order. 2. The repetition of elements in a set is meaningless. 3. A set may contain a Finite or Infinite number of elements. **KEY** **i.** '∈' = belongs to **ii.** ':' *or* '|' = such that **iii.** '⟹' = which implies
Description of Sets	**ROSTER / BRACES { }** Described by listing elements, separated by commas, within braces { }. *E.g. A = {a, e, i, o, u}* **SET BUILDER / ALGEBRAIC / RULE / PROPERTY** Described by a property P(x) of its elements x, such that {x : P(x) holds}. *E.g. A = {x : x is a of vowel in alphabets}*
Subset & Superset	If every element of a set is also an element of another set, the former is said to be the subset of the latter & the latter is said to be superset of the former. *E.g. P = {a, b, c} & Q = {a, b, c, d, e}, then we can say P is a subset of Q, or P ⊂ Q & Q is a superset of P, or Q ⊃ P.* A set containing 'n' elements has **2^n subsets** and **$2^n – 1$ proper subsets**. *E.g. A set containing 3 elements has 2^3 subsets and 2^3-1 proper subsets.*
Proper Subset	If a set is a Subset of another set, but not equal to another set, it is called Proper Subset. *E.g. {2, 3} is a Proper Subset of {2. 3. 5}, {1, 2} is not a Subset of {2. 3. 5}.*
Power Set	The collection of all possible subsets of set A. **It is denoted by P(A)**. *E.g. A = {1,2,3}, then P(A) = {{1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3}, Φ}*
Empty / Null / Void Set	▪ It contains no elements. ▪ It is usually denoted by **{} or Φ (phi)** ▪ A set which has at least 1 element is called **non-empty set**. ▪ It has no Proper Subset. ▪ Null set is a subset of every set. *E.g. A = {x : x is a Prime No. between 32 & 36}*
Cardinal No. of a Set	Total number of elements present in set A is called as its Cardinal No. It is denoted by **n(A)**. *E.g. A = {a,e,d,f}, then n(A) = 4*
	Equivalent Set ▪ Two finite sets A & B are said to be equivalent if **n(A) = n(B).** ▪ **It is denoted by A ↔ B or A ⇔ B.** *E.g. A = {1,2,3}. B = {a,b,c}, then A ↔ B* **Equal Set** ▪ Two finite sets A & B are said to be equal if Cardinal No. as well as elements are same. ▪ **It is denoted by A=B.** *E.g. A = {1,2,3}. B = {1,2,3}, then A=B* ***All equal sets are equivalent but all equivalent sets need not be equal.***
Singleton Set	A set containing a single element is known as Singleton Set. *E.g. A = {q}*
Difference of a Set	▪ A set containing all the elements that are in A but not in B is known as difference of sets A and B. ▪ **It is denoted by A – B or A ~ B.** ▪ **A – B = {x:x ∈ A & x ∉ B}** *E.g. A = {1,2,3,4,5}, B = {3,4,5} then A – B = {1,2}*
Universal Set	A set which contains all sets under consideration in a particular problem. **It is denoted by S.**
Complement of a Set	A set which contains all elements in S but not in A. **It is denoted by A^c or A'.** **A^c or A' = S – A = {x:x ∈ S & x ∉ A}.** ***DE MORGAN'S LAWS*** (A∪B)' = (A'∩B') (A∩B)' = (A'∪B')
Union of Sets	It is written as A∪B contain all the elements which are either in A or B or Both.
Intersection of Sets	It is written as A∩B contain those elements which are both in A & B.
Disjoint Sets	Two sets are said to be disjoint if there is no common element between them. **i.e. A∩B = Φ**

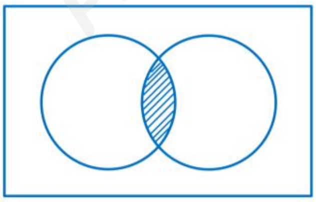
CONCEPT 02 : VENN DIAGRAM

UNION OF SETS



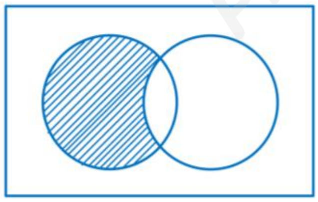
A∪B

INTERSECTION OF SETS



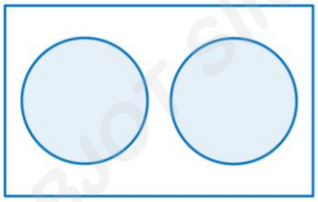
A∩B

DIFFERENCE OF SETS



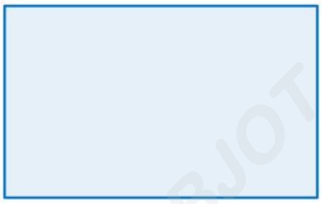
A – B

DISJOINT SET



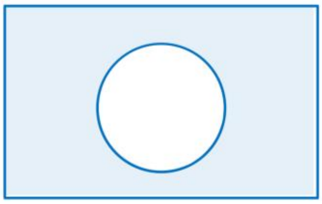
A∩B = Φ

UNIVERSAL SET



S

COMPLEMENT OF A SET



A'

CONCEPT 03 : PROPERTIES ON CARDINAL NUMBERS

A. n(A∪B) = n(A) + n(B) – n(A∩B)

B. n(A∪B) = n(A) + n(B) {A,B are Disjoint Non-Void Sets}

C. n(A – B) = n(A) – n(A∩B) = n(A∪B) - n(B)

D. n(A∪B∪C) = n(A) + n(B) + n(C) - n(A∩B) - n(B∩C) - n(A∩C) - n(A∩B∩C)
Note : *If A, B and C are Disjoint Sets, then n(A∪B∪C) = n(A) + n(B) + n(C)*

E. n(A∩B)' = n(A'∪B') = n(S) - n(A∩B)

F. n(A∪B)' = n(A'∩B') = n(S) - n(A∪B)

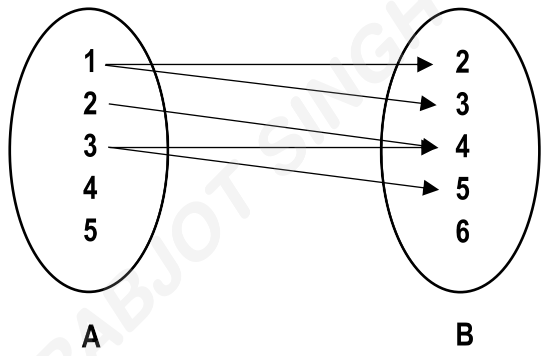
RELATIONS

CONCEPT 04 : PRODUCT OF SETS

Ordered Pairs	Two elements a & b, listed in a specific order, form an ordered pair, **denoted by (a,b)**.
Cartesian Product	▪ If A & B are two **non-empty** sets, then the set of all the ordered pairs (a,b) such that **a ∈ A & b ∈ B**, is called the Cartesian Product of A & B. **It is denoted by A x B**. *E.g. (a) A = {1, 2, 3} and B = {4, 5}, then A x B = { (1,4), (1,5), (2,4), (2,5), (3,4), (3,5)}* *(b) If A x B = {(3,2), (3,4), (5,2), (5,4)}, then A = {3, 5} and B = {2, 4}* ▪ **n(A x B) = n(A) x n(B)** *E.g. A = {1, 2, 3} and B = {4, 5}, n(A) = 3, n(B) = 2, then n(A x B) = 3 x 2 = 6.*

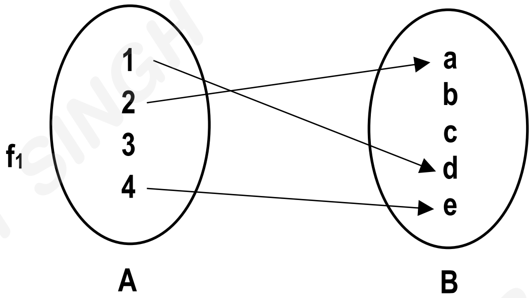
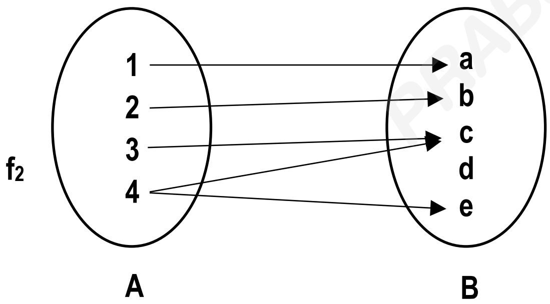
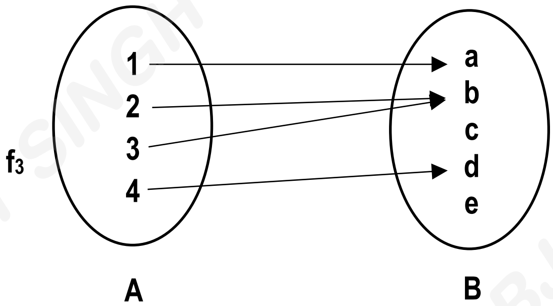
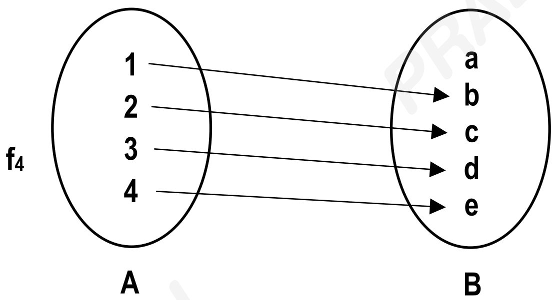
CONCEPT 05 : RELATIONS

Meaning	▪ Let A & B be two sets. Then a relation R from A to B is a subset of A x B. T Thus, R is a relation from **A to B ⇔ R ⊆ A x B**. ▪ If (a,b) ∈ R, then we write aRb, read as a is related to b by the relation R. ▪ If (a,b) ∉ R, then we write aR̸b, read as a is not related to b by the relation R.
Total No. of Relations	n(A) = x, n(B) = y, then Total No. of Relations = $\mathbf{2^{xy}}$ *E.g. A = {1, 2, 3} and B = {4, 5}, then Total No. of Relations =* $2^{3x2} = 2^6 = 64$
Domain	Let R be a relation from a set A to a set B. ▪ Set of all first components of the ordered pairs belonging to R : Domain of R ▪ Set of all second components of the ordered pairs belonging to R : Range of R ***Author's Note : Think of Domain as Input & Range as Output.***

Representation of Relation	**Roster Form**	If R is a relation from A = {-1, 0, 1} to B = {0, 1} by the rule aRb ⇔ $a^2 = b$. Then, 0R0, -1R1, 1R1. So, **R in Roster Form will be, R = {(-1,1), (0,0), (1,1)}**.
	Set-Builder Form	If A = {1, 2, 3} and B = {1, 1/2, 1/3} and R is a relation from A to B given by R = {(1,1), (2, 1/2), (3, 1/3)}. Then, **R in Set-Builder Form will be, R = {(a,b) : a ∈ A, b ∈ B and b = 1/a}**.
	Arrow Diagram	R = {(1,2), (2,4), (3,5), (1,3), (3,4)} from set A = {1, 2, 3, 4, 5} to B = {2, 3, 4, 5, 6} can be represented by **Arrow Diagram** as : 
Types of Relations	**Inverse Relation**	Let A & B be two sets and let R be a relation from A to B, then the Inverse of R, denoted by R^{-1} is a relation from B to A & is defined by **R^{-1} = {(b,a) : (a,b) ∈ R}**. *E.g. Let A = {1, 2, 3} and B = {a, b, c, d}, R = {(1,a), (1,c), (2,d), (2,c)}, then R^{-1} = {(a,1), (c,1), (d,2), (c,2)}.*
	Identity Relation	An Identity Relation on set A, denoted by I_A, means every element of A is related to itself only i.e. **I_A = {(a,a) : a ∈ A}**. *E.g. Let A = {1, 2, 3}, then I_A = {(1,1), (2,2), (3,3)} is an Identity Relation but I_A = {(1,1), (2,2)}, & I_A = {(1,1), (2,2), (3,3), (1,3)} are not Identity Relation.*
	Reflexive Relation	A relation R on a set A is said to be Reflexive every element of A is at least related to itself. *E.g. Let A = {1, 2, 3}, then R = {(1,1), (2,2), (3,3), (1,2), (2,1)} is Reflexive, but R = {(1,1), (3,3), (2,1), (3,2)} is not Reflexive.* ***All Identity Relations are Reflexive Relations, but all Reflexive Relations are not Identity Relations.***
	Symmetric Relation	A relation R on a set A is said to be Symmetric, iff **(a,b) ∈ R ⟹ (b,a) ∈ R**. *E.g. Let A = {1, 2, 3}, then* *(a) R_1 = {(1,2), (2,1), (2,2), (3,1), (1,3)} is a Symmetric Relation.* *(b) R_2 = {(1,1), (2,2), (3,1)} is not a Symmetric Relation.*
	Transitive Relation	Let A be any set. A relation R on set A is said to be transitive iff, **(a,b) ∈ R, (b,c) ∈ R ⟹ (a,c) ∈ R**.
	Equivalence Relation	A relation which is Reflexive, Symmetric and Transitive. *E.g. The relation "is parallel to" on the set S of all straight lines in a plane.* *(a) Reflexive, since a ll a for a ∈ R.* *(b) Symmetric, since a ll b ⟹ b ll a* *(c) Transitive, since a ll b, b ll c ⟹ a ll c*

FUNCTIONS

Meaning	Let A & B be two non-empty sets. A relation 'f' from A to B is called a Function from A to B, if **(a) for each $a \in A$, there exists $b \in B$, such that $(a,b) \in f$** **(b) $(a,b) \in f$ & $(a,c) \in f \Longrightarrow b = c$**
Understanding of Functions	Let A = {1, 2, 3, 4} and B = {a, b, c, d, e} and let f_1, f_2, f_3 & f_4 be relations from A to B as illustrated under :

	Function?	Explanation
	No	$3 \in A$ which is not associated to any element of B.
	No	$4 \in A$ is associated to more than 1 elements of B.
	Yes	Each element of A have a unique image in B.
	Yes	Each element of A have a unique image in B.

Author's Note

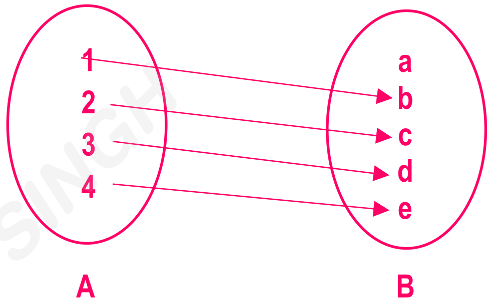
Multiple elements of A can have same image but they cannot have multiple images.

Domain, Co-Domain & Range

Let f : A → B

- Set A is known as **Domain.**
- Set B is known as **Co-Domain.**
- Set of all images of elements of A is known as **Range / Image** of Set A and is denoted by f(A).

E.g.

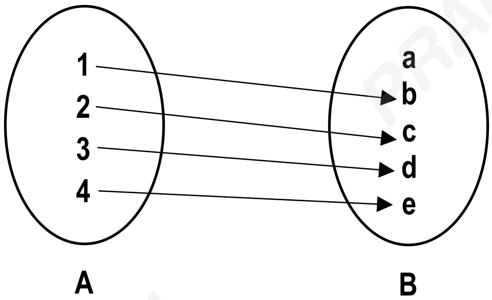


In this example,

Domain	*A = {1, 2, 3, 4}*
Co-Domain	*B = {a, b, c, d, e}*
Range	*f(A) = {b, c, d, e}*

TYPES OF FUNCTIONS

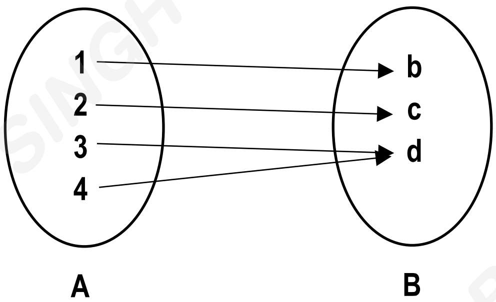
One-One [Injective] Function



Let f : A → B

If different elements in A have different images in B.

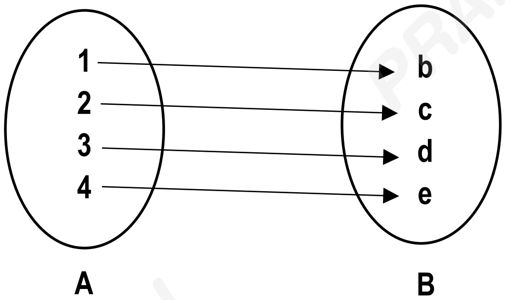
Onto [Surjective] Function



Let f : A → B

If every element in B has atleast one pre-image in A.

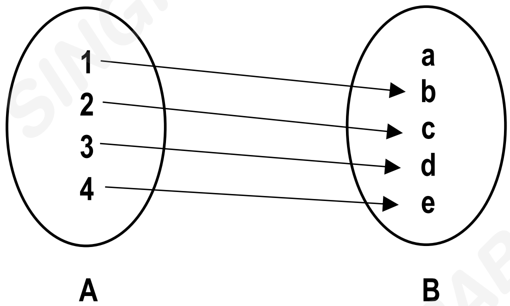
One-One Onto [Bijective] Function



Let f : A → B

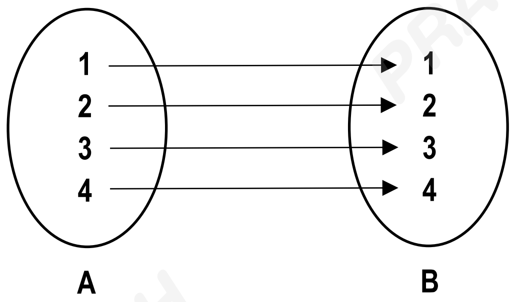
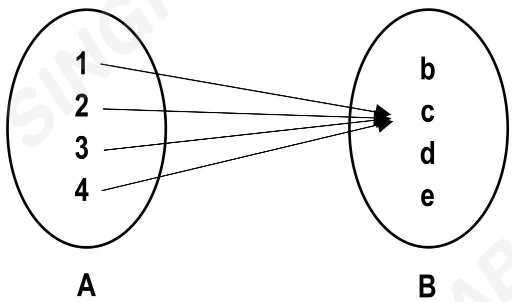
A One-One & Onto Function is known as Bijective Function (Every element in B has exactly one pre-image in A).

Into Function



Let f : A → B

There exists even a single element in B having no pre-image in A.

Identity Function	 The function that associates each real number to itself is an Identity Function & is denoted by I. **Domain =** R, **Range =** R
Constant Function	 **Let f : A → B** All elements of A have the same image in B. **Domain =** R, **Range =** Singleton Set
Equal Function	Two Functions f & g are said to be equal iff, **(a) Domain of f = Domain of g** **(b) Co-Domain of f = Co-Domain of g** **(c) f(x) = g(x), for every x ∈ to their common domain**
Inverse Function	Let f : A → B, such that f(x) = y. Then, **f^{-1} : B → A, will be f(y) = x.** ***A function is invertible iff, f is one-one onto.***
Composite Function	**Composite functions** are when the output of one function is used as the input of another. If we have a function f and another function g, the function fog(x), said as "f of g of x" is the composition of the two functions. *E.g. $f(x) = x + 3$, $g(x) = x^2$, then $fog(x) = f(g(x)) = f(x^2) = (x^2) + 3$*

07B. LIMITS & CONTINUITY

LIMITS

Meaning

- It is defined as a value that a function approaches the output for the given input values.
- It is important in calculus and used to define integrals, derivatives, and continuity.

Theorems on Limits

Let $\lim\limits_{x\to a} f(x) = p$ & $\lim\limits_{x\to a} g(x) = q$, where f(x) & g(x) are functions of x

THEOREMS	*EXAMPLES {f(x) = x² & g(x) = x}*
1. $\lim\limits_{x\to a}\{f(x)\pm g(x)\} = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = p \pm q$	$\lim\limits_{x\to 3}\{f(x)\pm g(x)\} = x^2 \pm x = 3^2 \pm 3$
2. $\lim\limits_{x\to a}\{f(x)\cdot g(x)\} = \lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x) = p\cdot q$	$\lim\limits_{x\to 3}\{f(x)\cdot g(x)\} = x^2\cdot x = 3^2\cdot 3$
3. $\lim\limits_{x\to a}\{f(x) / g(x)\} = \lim\limits_{x\to a} f(x) / \lim\limits_{x\to a} g(x) = p / q$	$\lim\limits_{x\to 3}\{f(x) / g(x)\} = x^2 / x = 3^2 / 3$
4. $\lim\limits_{x\to a} c = c$, where c is a constant	$\lim\limits_{x\to 3} 7 = 7$
5. $\lim\limits_{x\to a} cf(x) = c \lim\limits_{x\to a} f(x)$	$\lim\limits_{x\to a} 7f(x) = 7\cdot x^2 = 7\cdot 3^2$
6. $\lim\limits_{x\to a} F\{f(x)\} = F\{\lim\limits_{x\to a} f(x)\} = F(p)$	
7. $\lim\limits_{x\to 0}\frac{1}{x}$ does not exist.	

Methods of solving Limits

WHICH METHOD TO USE?

Does Denominator comes 0 after Substitution?

NO	*YES*	
Direct Substitution [E.g.1]	**Factorisation**	Solve it till 0/0 form is eliminated by Simplification *[E.g.2]*
	Rationalisation	Solve it till 0/0 form is eliminated by Rationalisation *[E.g.3]*
	Standard Limits **Algebraic Limits at Infinity**	Refer Standard Limits given below

E.g.1 $\lim\limits_{x\to 3}\frac{x^2+3}{x} = \frac{3^2+3}{3} = 4$

Explanation : *Direct Substitution Method will be followed as Denominator ≠ 0.*

E.g.2 $\lim\limits_{x\to 3}\frac{x^2-9}{x-3} = \lim\limits_{x\to 3}\frac{(x+3)(x-3)}{x-3} = \lim\limits_{x\to 3}(x+3) = 3+3 = 6$

Explanation : *Factorised till Denominator = 0 situation is eliminated and then x is substituted.*

E.g.3 $\lim\limits_{x\to 3}\frac{x-3}{\sqrt{x}-\sqrt{3}} = \lim\limits_{x\to 3}\frac{x-3}{\sqrt{x}-\sqrt{3}}\cdot\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} = \lim\limits_{x\to 3}\frac{(x-3)\cdot(\sqrt{x}+\sqrt{3})}{x-3} = \lim\limits_{x\to 3}(\sqrt{x}+\sqrt{3}) = 2\sqrt{3}$

Explanation : *Rationalised till Denominator = 0 situation is eliminated and then x is substituted.*

Standard Limits

- $\lim\limits_{x\to 0}\frac{(e^x-1)}{x} = 1$
- $\lim\limits_{x\to 0}\frac{(a^x-1)}{x} = \log_e a,\ (a>0)$
- $\lim\limits_{x\to 0}\frac{\log(1+x)}{x} = 1$
- $\lim\limits_{x\to 0}\frac{(x^n-a^n)}{x} = na^{n-1}$
- $\lim\limits_{x\to 0}\frac{(1-x)^n-1}{x} = n$
- $\lim\limits_{x\to a}\left(1+\frac{1}{x}\right)^a = \lim\limits_{x\to 0}\frac{(1+x)^{\frac{1}{x}}}{x} = e$

Algebraic Limits till ∞

- $\lim\limits_{x\to\infty}\frac{c}{x^n} = 0,\ n>0$

CONTINUITY

A function f(x) is said to be continuous at x = a iff,

(a) f(x) is defined at x = a

(b) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

(c) $\lim_{x \to a} f(x) = f(a)$

- In condition (b), both left handed and right handed limits are equal.
- In condition (c), limiting value of the function must be equal to its function value at x = a.

OTHER POINTS

(a) The sum, difference & product of two continuous functions is a continuous function. This property holds good for any finite number of functions.

(b) The quotient of two continuous functions is continuous function provided the denominator ≠ 0.

08. BASIC CONCEPTS OF DIFFERENTIAL & INTEGRAL CALCULUS

DIFFERENTIAL CALCULUS

Meaning	▪ Differentiation is the process of finding the derivative of a continuous function. ▪ It is defined as the ratio of change in the function corresponding to a small change in the independent variable. ▪ It is denoted as $\frac{dy}{dx}$ or $f'(x)$ or y' or y_1, where 'y' is function of x or f(x).	
Standard Formulas	1. $\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(x^7) = 7x^6$
	2. $\frac{d}{dx}(e^x) = e^x$	-
	3. $\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\frac{d}{dx}(e^{3x}) = 3e^{3x}$
	4. $\frac{d}{dx}(a^x) = a^x \cdot \log_e a$	$\frac{d}{dx}(7^x) = 7^x \cdot \log_e 7$
	5. $\frac{d}{dx}(\log x) = \frac{1}{x}$	$\frac{d}{dx}(\log 5) = \frac{1}{5}$
	6. $\frac{d}{dx}(\text{constant}) = 0$	$\frac{d}{dx}(7) = 0$
Laws for Differentiation	a. $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$	$\frac{d}{dx}(3x^2 + x) = \frac{d}{dx}3x^2 + \frac{d}{dx}x$
	b. $\frac{d}{dx}\{f(x) \cdot g(x)\} = f(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}f(x)$	$\frac{d}{dx}\{e^x \cdot x^2\} = e^x \cdot \frac{d}{dx}x^2 + x^2 \cdot \frac{d}{dx}e^x$
	c. $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x) \cdot \frac{d}{dx}f(x) - f(x) \cdot \frac{d}{dx}g(x)}{g(x)^2}$	$\frac{d}{dx}\frac{e^x}{x^2} = \frac{x^2 \cdot \frac{d}{dx}e^x - e^x \cdot \frac{d}{dx}x^2}{(x^2)^2}$
	d. $\frac{d}{dx}c \cdot f(x) = c \cdot \frac{d}{dx}\{f(x)\}$	$\frac{d}{dx}(7x) = 7\frac{d}{dx}(x)$
Higher Order Derivatives	Let $y = f(x) = x^4 + 5x^3 + 2x^2 + 9$, then, $\frac{dy}{dx} = \frac{d}{dx}f(x) = 4x^3 + 15x^2 + 4x = f'(x)$ Since f(x) is a function of x it can be differentiated again. Thus $\frac{d}{dx}(\frac{dy}{dx}) = f''(x) = \frac{d}{dx}(4x^3 + 15x^2 + 4x) = 12x^2 + 30x + 4$ ▪ $\frac{d}{dx}(\frac{dy}{dx})$ is written as $\frac{d^2y}{dx^2}$ (read as d two y by dx square) and is called the **Second Derivative of y with respect to x [f''(x)]** while $\frac{dy}{dx}$ is called the **First Derivative [f'(x)]**. ▪ Again the second derivative here being a function of x can be differentiated again and $\frac{d}{dx}(\frac{d^2y}{dx^2}) = f'''(x) = 24x + 30$.	

Types of Differentiation

FUNCTION OF FUNCTION [CHAIN RULE]

If we differentiate log $(1 + x^2)$ w.r.t x, then

Let $y = \log(1 + x^2)$

2nd Function

$\log\,(1 + x^2)$

1st Function

$\frac{dy}{dx} = \frac{d}{dx}(1 + x^2) \cdot \frac{d}{dx}\{\log(1 + x^2)\}$, using formulae

$\frac{dy}{dx} = (0 + 2x) \cdot \frac{1}{1 + x^2}$

$\mathbf{\frac{dy}{dx} = \frac{2x}{1 + x^2}}$

IMPLICIT FUNCTION

Let us consider $x^2y^2 + y = 0$, where y cannot be directly defined as a function of x.

Now, we'll differentiate both sides w.r.t x

$\frac{d}{dx}(x^2y^2 + y) = \frac{d}{dx}(0)$, **[Using Laws (a)&(b)]**

$\Rightarrow x^2 \frac{d}{dx}(y^2) + y^2 \frac{d}{dx}(x^2) + \frac{d}{dx}(y) = 0$

Differentiate using Chain Rule

$\Rightarrow x^2 \cdot 2y \frac{dy}{dx} + y^2 \cdot 2x + \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx}(2x^2y + 1) = -2xy^2 \Rightarrow \mathbf{\frac{dy}{dx} = \frac{-2xy^2}{2x^2y + 1}}$

PARAMETRIC FUNCTION

When both the variables x & y are expressed in terms of a parameter (a third variable), the involved equations are called **Parametric Functions.**

Let us consider, $x = at^3$ & $y = \frac{a}{t^3}$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ $\quad \frac{dx}{dt} = 3at^2$ $\quad \frac{dy}{dt} = \frac{-3a}{t^4}$

$\Rightarrow \frac{dy}{dx} = \frac{-3a}{t^4} \cdot \frac{1}{3at^2}$

$\Rightarrow \mathbf{\frac{dy}{dx} = \frac{-1}{t^6}}$

LOGARITHMIC FUNCTION

The process of finding out derivative by taking logarithm in the first instance is called **Logarithmic Function.**

Let us consider, $y = x^x$, taking Log both sides

$\Rightarrow \log y = \log x^x$

$\Rightarrow \log y = x \log x$

Now differentiating w.r.t. x both sides we get,

$\Rightarrow \frac{d}{dx}(\log y) = \frac{d}{dx} x \log x$ **[Using Law (b)]**

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 1 + \log x$

$\Rightarrow \frac{dy}{dx} = y(1 + \log x) \Rightarrow \mathbf{\frac{dy}{dx} = x^x(1 + \log x)}$

Author's Note : We will take Log only in cases like x^x, x^y, y^y, y^x etc.

Application of Derivatives

COST FUNCTION

C(x) = V(x) + F(x)

$\mathbf{AC} = \frac{C(x)}{x}$ $\quad \mathbf{AVC} = \frac{V(x)}{x}$ $\quad \mathbf{AFC} = \frac{F(x)}{x}$

Marginal Cost (MC) = $\frac{d}{dx}[C(x)]$

Key :

- ***C(x)** = Total Cost*
- ***V(x)** = Variable Cost*
- ***F(x)** = Fixed Cost*
- ***AC** = Average Cost*
- ***AVC** = Average Variable Cost*
- ***AFC** = Average Fixed Cost*

REVENUE FUNCTION

R(x) = Price (P) · No. of Units (x) = Px

- **Marginal Revenue (MR) =** $\frac{d}{dx}[R(x)]$
- **Profit Function [P(x)] =** R(x) – C(x)
- **Marginal Profit =** $\frac{d}{dx}[P(x)]$

MARGINAL PROPENSITY TO CONSUME (MPC)

The rate of Change of Consumption (C) per unit Change in Income (Y) i.e. $\frac{dC}{dY}$.

MARGINAL PROPENSITY TO CONSUME (MPS)

The rate of Change of Saving (S) per unit Change in Income (Y) i.e. $\frac{dS}{dY}$.

INDEFINITE INTEGRATION

Meaning

Integration is the reverse process of Differentiation and is denoted by $\int$.

Standard Formulas

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$, where $n \neq -1$
2. $\int dx = x + c$
3. $\int e^x\,dx = e^x + c$
4. $\int e^{ax}\,dx = \dfrac{e^{ax}}{a} + c$
5. $\int \dfrac{dx}{x} = \log x + c$
6. $\int a^x\,dx = \dfrac{a^x}{\log_e a} + c$
7. $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a}\log\dfrac{x-a}{x+a} + c$
8. $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \log\left|x + \sqrt{x^2 \pm a^2}\right| + c$
9. $\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a}\log\dfrac{a+x}{a-x} + c$
10. $\int e^x\{f(x) + f'(x)\}\,dx = e^x f(x) + c$
11. $\int \sqrt{x^2 \pm a^2}\,dx = \dfrac{x}{2}\sqrt{x^2 \pm a^2} \pm \dfrac{a^2}{2}\log\left|x + \sqrt{x^2 \pm a^2}\right| + c$
12. $\int \dfrac{f'(x)}{f(x)}\,dx = \log f(x) + c$

Author's Note : We add 'c' (constant of integration) in every sum, since differentiation of constant is always 0.

Elementary Rules

A. $\int a\,f(x)\,dx = a\int f(x)\,dx$, where a is constant

B. $\int \{f(x) \pm g(x)\}\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Methods for Integration

SUBSTITUTION METHOD

In this method, any given integral is transformed into a simple form of integral by substituting the independent variable by others.

Let us consider, $y = \int (2x+3)^7 dx$.

$\Rightarrow$ Substituting $t = 2x + 3$, $\dfrac{dt}{dx} = 2 \Rightarrow \dfrac{dt}{2} = dx$

$\Rightarrow y = \int t^7 \dfrac{dt}{2} = \dfrac{1}{2}\int t^7 dt$

$\Rightarrow y = \dfrac{1}{2}\cdot\dfrac{t^8}{8} + c = \dfrac{t^8}{16} + c$

$\Rightarrow$ **$y = \dfrac{(2x+3)^8}{16} + c$**

INTEGRATION BY PARTS

$$\int u\,v\,dx = u\int v\,dx - \int\left\{\frac{d(u)}{dx}\int v\,dx\right\}dx$$

Second Term → v; First Term → u

Let us consider, $y = \int x \log x\,dx$

On Integrating by Parts,

$\Rightarrow y = \log x\int x\,dx - \int\left\{\dfrac{d}{dx}(\log x)\int x\,dx\right\}dx$

$\Rightarrow y = \dfrac{x^2}{2}\log x - \int\left[\dfrac{1}{x}\cdot\dfrac{x^2}{2}\right]dx$

$\Rightarrow y = \dfrac{x^2}{2}\log x - \dfrac{1}{2}\int x\,dx$

$\Rightarrow$ **$y = \dfrac{x^2}{2}\log x - \dfrac{x^2}{4} + c$**

Order of Priority for taking First Term

- Logarithm (log)
- Algebra (x^n)
- Trigonometric Ratios [Not in Syllabus]
- Exponential Term (e^x)

PARTIAL FRACTIONS

A function of the form $\dfrac{f(x)}{g(x)}$, where degree of f(x) < degree of g(x) is called Proper, otherwise Improper.

$\dfrac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\dfrac{A}{x-a} + \dfrac{B}{x-b}$	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$
$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{x-a} + \dfrac{Bx+C}{x^2+bx+c}$	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a} + \dfrac{B}{(x-a)^2} + \dfrac{C}{x-b}$

DEFINITE INTEGRATION

Meaning	$\int_a^b f(x)\,dx = f(b) - f(a)$	Here, 'b' is called the Upper Limit and 'a' the Lower Limit of Integration.
Properties	1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$	2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
	3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, a<c<b	4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
	5. When f(x) = f(a + x), then $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$	
	6. $\int_a^b f(x)\,dx = 2\int_0^a f(x)\,dx$, if f(-x) = f(x); $= 0$, if f(-x) = -f(x)	
	Author's Note : We don't add the constant (c) in Definite Integration.	
Application	**Cost Function**	$C(x) = \int MC\,dx + k$, where MC = Marginal Cost, k = Fixed Cost
	Revenue Function	$R(x) = \int MR\,dx + k$. Also Demand Function $(p) = \frac{R(x)}{x}$.

09. NUMBER SERIES, CODING DECODING, ODD MAN OUT

CONCEPT 01 : POSITION OF ALPHABETS

In Forward Order & Backward Order

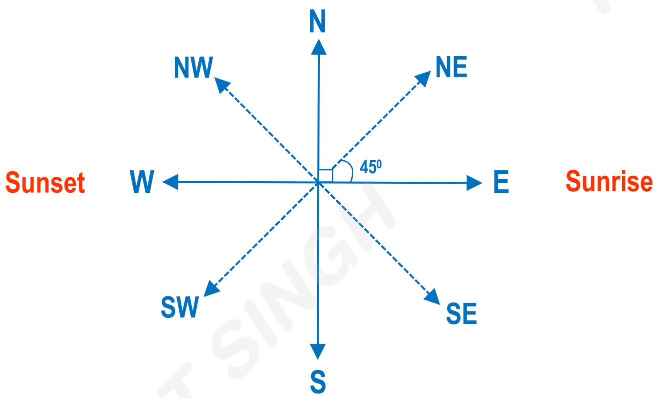
A		B		C		D		E	
1	26	2	25	3	24	4	23	5	22
F		G		H		I		J	
6	21	7	20	8	19	9	18	10	17
K		L		M		N		O	
11	16	12	15	13	14	14	13	15	12
P		Q		R		S		T	
16	11	17	10	18	9	19	8	20	7
U		V		W		X		Y	
21	6	22	5	23	4	24	3	25	2
Z		*Trick 01 : Backward = 27 - Forward*							
26	1	*Trick 02 : EJOTY are at multiples of 5*							

GENERAL PATTERNS ASKED IN EXAMINATION

- Constant Addition / Subtraction
- Constant Addition / Subtraction of multiples of a certain number
- Constant Addition / Subtraction of Odd / Even multiples of a certain number
- Constant Addition of Prime Numbers
- Constant Addition of Squares / Cubes of Consecutive Numbers
- Constant Multiplication / Division
- In some series, the pattern can be seen after calculating double difference.
- Some series contain two series with alternate numbers which contain different patterns.

10. DIRECTION SENSE TEST

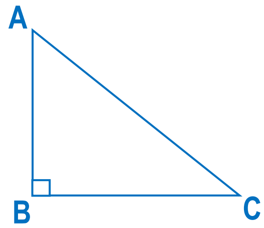
CONCEPT 02 : DIRECTIONS



CONCEPT 03 : SHORTEST DISTANCE

To find the shortest distance, we need to know about the Pythagoras Theorem :

Here,
AB = Perpendicular
BC = Base
AC = Hypotenuse



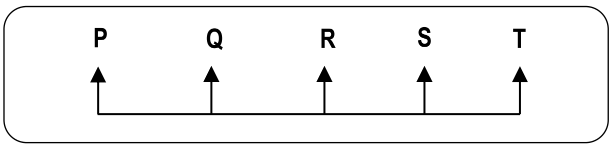
Now, by Pythagora's Theorem, shortest distance between Point A & Point C :

$$AC = \sqrt{(AB)^2 + (BC)^2}$$

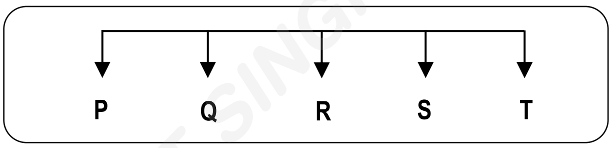
11. SEATING ARRANGEMENTS

CONCEPT 04 : LINEAR ARRANGEMENTS

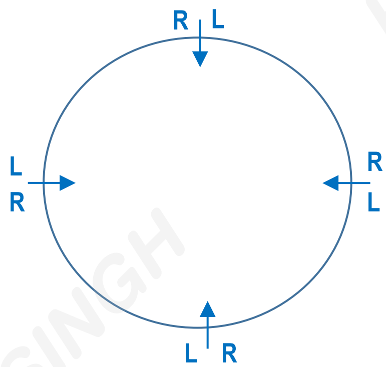
- When the direction of face is not clear :



- When the direction of the face is towards you :



CONCEPT 05 : CIRCULAR ARRANGEMENTS



Author's Note : Students are advised to use their imagination & logic to make the diagram where they are sitting & facing away from the centre.

12. BLOOD RELATIONS

CONCEPT 06 : BASIC RELATIONS

Father / Mother's	**Father**	Grandfather
	Mother	Grandmother
	Brother	Uncle
	Sister	Aunt
	Son	Brother / Self
	Daughter	Sister / Self

You	**Spouse's Brother**	Brother-in-Law
	Spouse's Sister	Sister-in-Law
	Son's Wife	Daughter-in-Law
	Daughter's Husband	Son-in-Law
	Brother's Wife	Sister-in-Law
	Sister's Husband	Brother-in-Law

Uncle / Aunt's	**Son / Daughter**	Cousin

Brother / Sister's	**Son**	Nephew
	Daughter	Niece

OTHER POINTS

- Paternal means relations from Father's Side.
- Maternal means relations from Mother's Side.

13A. STATISTICAL DESCRIPTION OF DATA

CONCEPT 01 : INTRODUCTION TO STATISTICS

Definition	**Singular Noun**	The **scientific method** that is employed for collecting, analysing & presenting data, leading finally to drawing statistical inferences about important characteristics. It is **Science of Counting** or **Science of Averages**.
	Plural Noun	**Data, qualitative as well as quantitative**, that are collected, usually with a view of having statistical analysis.
History of Statistics	▪ **Origin of the word Statistics** : This is a debatable topic but various theories are as follows : **(a) Latin Word :** Status **(c) German Word :** Statistik **(b) Italian Word :** Statista **(d) French Word :** Statistique ▪ The first census was conducted by Pharaoh in Egypt during 300BC to 2000BC. ▪ Kautilya (Chanakya) kept a record of births & deaths and some other precious information in his book 'Arthashastra' during Chandragupta Maurya's reign in 4th Century BC. ▪ Statistical records on Agriculture are also found in Ain-i-Akbari by Abul Fazl in 16th Century AD.	
Application	**Economics**	▪ **Econometrics :** Branch of Economics that interacts with Statistics ▪ Time Series Analysis, Index Numbers, Regression Analysis, Demand Analysis etc. are some overlapping areas.
	Business Management	**Statistical Decision Theory** focuses on the analysis of complicated business strategies with a list of alternatives.
	Commerce & Industry	Data on previous sales, raw materials, wages etc. are collected, analysed & experts are consulted in order to maximise profits.
Limitations	(a) **It deals with aggregates.** An individual has no significance. (b) **It is concerned with quantitative data.** However, qualitative data can also be converted into quantitative data by assigning a numerical value. (c) **Future projections are possible under specific set of conditions.** If any of these conditions are violated, projections are likely to be incorrect. (d) **The theory of statistical inferences is built upon random sampling.** If the rules of random sampling are not strictly adhered to, the conclusion drawn based on these unrepresentative samples would be erroneous.	

CONCEPT 02 : COLLECTION OF DATA

What is Data?

- Data is a quantitative information about some particular characteristic(s) under consideration.
- A **Variable** is a measurable quantity i.e. quantitative information.

Discrete Variable	If it can assume only finite or countably infinite no. of isolated values. *Examples :* *(a) No. of Petals in a Flower* *(b) No. of Misprints in a Book* *(c) No. of Road Accidents in a Locality* *(d) Annual Income of a Person* *(e) Marks of a Student* *(f) Distribution of Shares* *(g) Salary of a Person*
Continuous Variable	If it can assume any value from a given interval. *Examples :* *(a) Height, Weight, Age of a Person* *(b) Sales / Turnover of a Company* *(c) Distribution of Profits of a Company*

- An **Attribute** is a qualitative characteristic.

Examples :

(a) Gender of a Baby,   *(c) Colour of a Person,*

(b) Nationality of a Person   *(d) Drinking Habit of a Person, etc.*

<table>
<tr><td rowspan="2">Classification of Data</td><td colspan="2">PRIMARY DATA</td><td>SECONDARY DATA</td></tr>
<tr><td colspan="2">The data which is collected for the first time by an investigator or an agency.</td><td>Data, already collected, used by a different person or agency.</td></tr>
<tr><td rowspan="9">Methods of Collecting Primary Data</td><td colspan="3">[A] INTERVIEW METHOD</td></tr>
<tr><td>Personal Interview</td><td colspan="2">▪ The investigator meets the respondents directly & collects the required information then & there, from them.
▪ In case of a Natural Calamity, this method can be quick & accurate.</td></tr>
<tr><td>Indirect Interview</td><td colspan="2">▪ The investigator collects necessary information from the persons associated with the problems.
▪ If there are some practical problems in reaching the respondents directly (like in a rail accident), then this method can be used.</td></tr>
<tr><td>Telephonic Interview</td><td colspan="2">▪ The information can be gathered by researcher by contacting the interviewee over the phone.
▪ It is a quick & non-expensive way to collect data.
▪ It is less consistent method, but has a wide coverage.
▪ Non-Responses are maximum in this method.</td></tr>
<tr><td colspan="3">[B] QUESTIONNAIRE METHOD</td></tr>
<tr><td>Mailed Questionnaire</td><td colspan="2">▪ It involves framing of a well-drafted & soundly-sequenced questionnaire covering all the important aspects of the problem & sending them to the respondents with pre-paid stamp & necessary guidelines.
▪ It has a wide coverage but non responses are maximum in this method.</td></tr>
<tr><td>Questionnaire by Enumerator</td><td colspan="2">▪ Enumerators collect information directly by interviewing the persons having information by explaining the questions.
▪ It is used for larger enquiries from persons who are being surveyed.</td></tr>
<tr><td colspan="3">[C] OBSERVATION METHOD</td></tr>
<tr><td colspan="3">▪ In this method data is collected by direct observation or using an instrument (like height or weight of a group of students).
▪ It is time consuming & laborious method having small coverage.
▪ It is the best method for data collection.</td></tr>
<tr><td>Sources of Secondary Data</td><td colspan="3">1. International Sources : WHO, ILO, IMF, World Bank etc.
2. Government Sources : Statistical Abstract by CSO, Indian Agricultural Statistics etc.
3. Private & Quasi Govt. Sources : ISI, ICAR, NCERT etc.
4. Unpublished Sources of various research institutes, researchers etc.</td></tr>
<tr><td>Scrutiny of Data</td><td colspan="3">▪ Since the statistical analysis are made only on the basis of data, it is necessary to check whether the data under consideration are accurate as well as consistent.
▪ No hard & fast rules can be recommended for scrutiny of data. One must apply his intelligence, patience & experience while scrutinising the given information.
▪ If two or more series of related data are given, they may be checked for Internal Consistency.
E.g. If data for Population, Area & Density for some places are given, then we may verify that they are internally consistent by examining whether the relation, Density = Population / Area holds.
▪ A good enumerator can also detect whether the returns submitted by some enumerators are exactly of the same type, thereby implying the lack of seriousness on the part of enumerators.
▪ The bias of enumerators may also be reflected by the returns submitted by him.
Rectification : This type of errors can be rectified by asking the enumerator(s) to collect the data of disputed cases, once again.</td></tr>
</table>

CONCEPT 03 : CLASSIFICATION / ORGANISATION OF DATA

Definition

The process of arranging data on the basis of the characteristic under consideration into a number of groups or classes according to the similarities of the observations.

Characteristic	Type of Data	*Example*
NON-FREQUENCY DATA		
Time Points / Intervals	**Chronological / Temporal / Time Series Data**	*No. of students appeared in CA Foundation in last 20 years.*
Region	**Spatial Series / Geographical Data**	*No. of students appeared in CA Foundation in 2025 in accordance with different states.*
FREQUENCY DATA		
Variable	**Quantitative / Cardinal Data**	*Height, Weight, Profits*
Attribute	**Qualitative / Ordinal Data**	*Gender, Nationality*

Advantages

(a) It puts data in a neat, precise and condensed form so that it is **easily understood & interpreted**.
(b) It makes **comparison possible** between various characteristics.
(c) **Statistical Analysis** is possible only for the classified data.
(d) It **eliminates unnecessary details** & makes data more readily understandable.

CONCEPT 04 : PRESENTATION OF DATA

4.1 TEXTUAL PRESENTATION OF DATA

Definition

- This method comprises presenting data with the help of a paragraph(s).
- The official reports of enquiry commissions are usually made through this method.

Example

"In 2024, out of a total of five thousand workers of Roy Enamel Factory, four thousand and two hundred were members of a trade union. The total number of female workers was eight hundred & six per cent of total members were members of the Trade Union.

In 2025, the number of workers belonging to the trade union was increased by twenty per cent as compared to 2024 of which four thousand and two hundred were male. The number of workers not belonging to trade union was nine hundred and fifty of which four hundred and fifty were females."

Advantages	Disadvantages
▪ Simplicity ▪ Observations with exact magnitude can be presented ▪ First step towards other modes of presentation	▪ Dull & Monotonous ▪ Comparison is not possible ▪ Cannot be recommended for manifold classification

4.2 TABULAR PRESENTATION / TABULATION

Definition

A systematic presentation of data with the help of statistical table having a number of rows & columns & complete with reference number, title, description of rows as well as columns and foot notes, if any.

Guidelines for Tabulation

Table Number
Title

<table>
<tr><td rowspan="3">Stub
(Row Heading)</td><td colspan="4">Caption (Column Heading)</td><td rowspan="3">Total (Rows)</td></tr>
<tr><td colspan="2">Sub-Head</td><td colspan="2">Sub-Head</td></tr>
<tr><td>Column Head</td><td>Column Head</td><td>Column Head</td><td>Column Head</td></tr>
<tr><td>⋮
Stub Entries
(Row Entries)
⋮</td><td colspan="4">Body</td><td></td></tr>
<tr><td>Total(Columns)</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Source :
Footnotes :

<table>
<tr><td></td><td>
1) Serial Number with a Self-Explanatory Title
2) Table should be divided into Caption, Box-Head (Entire Upper Part), Stub & Body.
3) Table should be well-balanced in length & breadth.
4) Data must be arranged in such a way that comparisons are facilitated.
5) Row Totals, Column Totals, Units of Measurement must be shown.
6) Data should be arranged intelligently and appealing to the eyes as far as possible.
7) Source of Data and Footnotes should be shown (if any).
<u>TERMINOLOGY</u>
(a) Box Head : The entire upper portion of the table which includes columns & sub-column numbers, unit(s) of measurement & caption.
(b) Caption : Upper Portion of the table describing columns & sub-columns.
(c) Stub : Left Portion of the table providing description of the rows.
</td></tr>
<tr><td>Example</td><td>
Table 13.1
Status of the workers of Roy Enamel Factory on the basis of their trade union membership for 2024 & 2025
<table>
<tr><th>Status</th><th colspan="3">Member</th><th colspan="3">Non-Member</th><th colspan="3">Total</th></tr>
<tr><th>Year</th><th>M</th><th>F</th><th>T</th><th>M</th><th>F</th><th>T</th><th>M</th><th>F</th><th>T</th></tr>
<tr><td>2024</td><td>3900</td><td>300</td><td>4200</td><td>300</td><td>500</td><td>800</td><td>4200</td><td>800</td><td>5000</td></tr>
<tr><td>2025</td><td>4200</td><td>840</td><td>5040</td><td>500</td><td>450</td><td>950</td><td>4700</td><td>1290</td><td>5990</td></tr>
</table>
Source :
Footnote : M stands for Male, F stands for Female & T stands for Total.
</td></tr>
<tr><td>Advantages</td><td>
▪ Facilitates comparison between rows & columns.
▪ Complicated data can also be represented using tabulation.
▪ It is a must for diagrammatic representation.
▪ It is a must for statistical analysis.
▪ It is the most accurate & best method of presentation of data.
</td></tr>
</table>

4.3 DIAGRAMMATIC PRESENTATION

<table>
<tr><td>Basics</td><td>
▪ This can be used for both educated & uneducated section of society, unlike the previous two.
▪ Any hidden trend can be noticed only in this mode of presentation.
▪ If there is a priority for accuracy, recommend tabulation. It is less accurate than tabulation.
▪ It is the most attractive method of presentation of data.
</td></tr>
<tr><td>Line Diagram /Historiagram</td><td>
It is used for Time Series Data.
(a) <u>If wide range of Fluctuations</u>
Logarithmic / Ratio Chart
(b) <u>Multiple Time Series Data</u>
▪ Same Unit : Multiple Line Chart
▪ Distinct Unit : Multiple Axis Chart
</td></tr>
<tr><td>Bar Diagram</td><td>
(a) <u>Time Series Data / Quantitative Data</u>
Vertical Bar Diagram
(b) <u>Spatial Data / Qualitative Data</u>
Horizontal Bar Diagram
(c) <u>For Comparing Related Series</u>
Multiple / Grouped Bar Diagram
(d) <u>For Representing Data into Parts</u>
Component / Sub-Divided Bar Diagram
(e) <u>For Comparing Different Components or Relating the Components to the Whole</u>
Divided / Percentage Bar Diagram
</td></tr>
<tr><td>Pie Chart</td><td>
▪ For Circular Presentation of Data & For Comparing Different Components or Relating the Components to the Whole.
▪ <u>Segment Angle</u>
Segment Value / Total Value x 360°
</td></tr>
</table>

CONCEPT 05 : FREQUENCY DISTRIBUTION

Frequency: No. of times a particular observation / class occurs.

Frequency Distribution: It is a statistical table that distributes the total frequency to a number of classes.

TYPES OF FREQUENCY DISTRIBUTION

DISCRETE / UNGROUPED / SIMPLE F.D.	GROUPED F.D.
When tabulation is done in respect of a Discrete Random Variable & frequency is assigned to each one of them.	When tabulation is done in respect of a Continuous Variable & frequency is assigned to a group of values & not individual values. **Types of Grouped Classification** (a) Non-Overlapping / Mutually Inclusive (b) Overlapping / Mutually Exclusive

Mutually Inclusive Classification

Class Interval	x
01 – 10	
11 – 20	
21 – 30	

CI includes Upper Class Limit. (10 will be counted in 01-10 CI)

Mutually Exclusive Classification

Class Interval	x
01 – 10	
10 – 20	
20 – 30	

CI excludes Upper Class Limit. (10 will be counted in 10-20 CI)

Basic Terminology

Class Limit (CL): The Minimum Value (Lower Class Limit) & Maximum Value (Upper Class Limit) of a Class Interval (CI).

Class Boundary (CB):
- ***Mutually Exclusive Classification :*** *Class Boundary = Class Limits.*
- ***Mutually Inclusive Classification :***
  Class Boundary will be calculated as follows :

LCB = LCL – D/2	***UCB = UCL + D/2***
LCB = 10 – ½ = 9.5	*UCB = 19 + ½ = 19.5*
9.5, 19.5, 29.5, 39.5 etc.	*19.5, 29.5, 39.5 etc.*

**D = Difference between UCL of given CI & LCL of next CI.*

Class Length: **Class Length / Width / Size =** UCB – LCB

Class Mark: **Mid-Point / Mid-Value =** $\frac{LCL + UCL}{2} = \frac{LCB + UCB}{2}$

Frequency Density: = Frequency of a CI / Class Length

Relative Frequency: = Class Frequency / Total Frequency [They add up to 1 (Unity)]

% Frequency: = Class Frequency / Total Frequency x 100 [They add up to 100%]

Note : *If question asks to calculate No. of Class Intervals, then follow these steps :*
- ***Step 01 :*** *Calculate Range (Largest Observation – Smallest Observation).*
- ***Step 02 :*** *No. of CI = Divide Range by Class Length required (Rounded Up).*

(Question might give 1+3.322logN as an option. Learn this if asked again.)

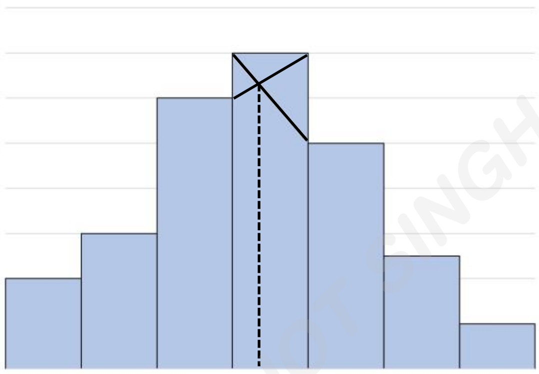
Cumulative Frequency

TYPES OF CUMULATIVE FREQUENCY

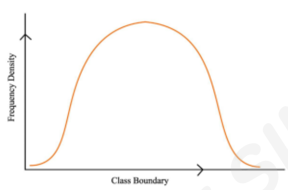
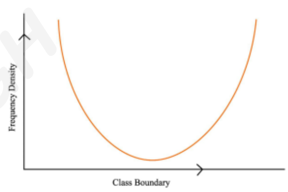
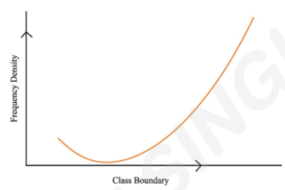
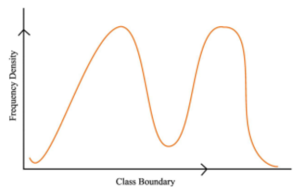
Class Interval	Frequency (f)	Less than	CF	More than	CF
10 – 15	**20**	Less than 15	20	More than 10	100 (80 + 20)
15 – 20	**32**	Less than 20	52 (20 + 32)	More than 15	80 (48 + 32)
20 – 25	**18**	Less than 25	70 (52 + 18)	More than 20	48 (30 + 18)
25 – 30	**30**	Less than 30	100 (70 + 30)	More than 25	30
	$\sum$**f = 100**				

For any Class Boundary, Less than CF + More than CF = $\sum f$

CONCEPT 06 : GRAPHICAL REPRESENTATION OF FREQUENCY DISTRIBUTION

Histogram (Area Diagram)	▪ It is a very **convenient way** to represent frequency distribution. ▪ Comparison among different CI's is possible. ▪ It is used to calculate **Mode**.  **MODE**
Frequency Polygon	▪ It is usually meant for Simple / Ungrouped Frequency Distribution. ▪ But it can also be used for Grouped Frequency Distribution, provided the width of the Class Intervals remains the same (Use Mid-Points). ▪ We can also obtain a Frequency Polygon starting with a Histogram, by adding the mid-points of the upper sides of the rectangles successively & then completing the figure by joining the two ends.
Ogives	▪ By plotting cumulative frequency against the respective class boundary, we get ogives. ▪ By plotting less than cumulative frequency, we get **Less than type Ogive**. ▪ By plotting more than cumulative frequency, we get **More than type Ogive**. ▪ They can be considered for obtaining **Quartiles** graphically. ▪ If a perpendicular is drawn from the point of intersection of the two ogives on the horizontal axis, then the x-value of this point gives us the value of **Median**. ▪ They can also be used for making **short term projections**.
Frequency Curve	▪ It is a limiting form of a Histogram or Frequency Polygon. ▪ It can be obtained by drawing a smooth & free hand curve through the mid points of the upper sides of the rectangles forming Histogram. ▪ Total Area is taken to be 1 (Unity). ▪ **X – Axis :** Class Boundary & **Y – Axis =** Frequency Density

TYPES OF FREQUENCY CURVE

Bell Shaped Curve	U-Shaped Curve	J-Shaped Curve	Mixed Curve
Frequency is maximum near central part & minimum near extremities. It is the most commonly used curve.	Frequency is minimum near the central part & maximum near extremities.	Starts with minimum frequency & reaches maximum at other extremity.	Combination of different curves.
			

13B. SAMPLING

CONCEPT 01 : INTRODUCTION TO SAMPLING

Population (Universe)

- It can be defined as the aggregate of all the units under consideration.
  E.g. Population of students enrolled for CA Foundation.
- The No. of Units belonging to a population is known as **Population Size (N)**.
- The study of every element of population is called **Census**.

TYPES OF POPULATION

Type		*Example*
Finite Population	Population containing finite no. of units.	*Population of students enrolled for CA Foundation*
Infinite Population	Population containing infinite or uncountable no. of units.	*Population of Stars, Mosquitos, Flowers, Insects*
Existent Population	Population consisting of real objects	*Population of a town*
Imaginary Population	Population that exists hypothetically	*Population of heads of a coin tossed infinitely.*

Sample

- It can be defined as a part of population so selected with a view to represent the population in all its characteristics.
- The units forming sample are known as **Sampling Units**.
- A detailed & complete list of all the Sampling Units is known as **Sampling Frame**.
- If a sample contains 'n' units, then 'n' is known as **Sample Size**.

Parameter

- It can be defined as a **characteristic of a population based on all the units of the population**.
- Statistical Inferences are drawn about population parameters based on the sample observations drawn from that population.

Statistic (T)

- It can be defined as a **statistical measure of sample observation** & it is a function of sample observations.

CONCEPT 02 : SAMPLE SURVEY

Meaning: It is the study of the unknown population on the basis of a proper representative sample drawn from it.

Principles of Sample Survey

Law of Statistical Regularity	If a sample of fairly large size is drawn from the population at random, then on an average the sample would possess the characteristics of that population.
Principle of Inertia	The results are more reliable, accurate & precise as the sample size increases, provided other factors are kept constant.
Principle of Optimisation	An optimum level of efficiency at a minimum cost or the maximum efficiency at a given level of cost can be achieved with the selection of an appropriate sampling design.
Principle of Validity	A sampling design is valid only if it is possible to obtain valid estimates & valid tests about population parameters (Only Probability Sampling).

Errors / Bias in Sample Survey

It can be defined as the deviation between the value of population parameter as obtained from a sample & its observed value.

TYPES OF ERRORS

[1] SAMPLING ERRORS

- Since only a part of the population is investigated in a sampling, every sampling design is subject to this type of errors.
- **Factors**
  (a) Due to Defective Sampling Design
  (b) Due to Substitution (of a Sampling Unit)
  (c) Due to Faulty Demarcation of Units
  (d) Due to wrong choice of Statistic
  (e) Variability in Population

[2] NON-SAMPLING ERROR

- This type of error happens in both Sampling & Complete Enumeration.
- **Factors :** Lapse of Memory, Preference for certain digits, ignorance, psychological factors like vanity, non-response on part of interviewees, wrong measurement of sampling units, communication gap, incomplete coverage.

CONCEPT 03 : TYPES OF SAMPLING

Types	PROBABILITY SAMPLING	NON PROBABILITY SAMPLING	MIXED SAMPLING
	1. Simple Random Sampling 2. Stratified Sampling 3. Multi-Stage Sampling	Purposive / Judgment Sampling	Systematic Sampling

Probability Sampling

In this there is always a fixed, pre-assigned probability for each member of the population to be a part of the sample taken from that population.

Simple Random Sampling

- Each unit of sample has an equal chance of being selected.
- It is very simple & effective method provided :
  - (a) The population is not very large.
  - (b) The sample size is not very small.
  - (c) The population under consideration is not heterogeneous.
- It is completely free from Sampler's Bias.
- All the tests of significance are based on the concept of Simple Random Sampling.

Stratified Sampling

- If the population is large & heterogeneous, we divide them into a number of sub-populations (strata) in such a way that there should be little variations among units in a stratum & maximum difference among different strata.
- If Simple Random Sampling is applied for drawing units from all strata, it is known as **Stratified Random Sampling**.
- Purpose of Stratified Sampling are :
  - (a) To make representation of all the sub-populations
  - (b) To provide an estimate of parameter not only for all the strata but also an overall estimate
  - (c) Reduction in variability and thereby an increase in precision
- **Types of Allocation of Sample Size**
  1. **Bowley's Allocation (Proportional Allocation)**
     When there is prior information that there is not much variation between the strata variances, we use Bowley's Allocation, where the sample sizes for different strata are taken proportional to the population sizes i.e. $n_i \propto N_i$.
  2. **Neyman's Allocation**
     When the strata variances differ significantly among themselves, we use Neyman's Allocation, where sample size vary jointly with population size & population standard deviation i.e. $n_i \propto N_i S_i$.

  Here,
  - → n_i = Sample Size for i^{th} stratum,
  - → N_i = Population Size
  - → S_i = Population Standard Deviation
- It is not advisable if :
  - (a) Population is not large.
  - (b) Some prior information is not available.
  - (c) There is not much heterogeneity among the units of population.

Multi Stage Sampling

- In this type of sampling design, sampling is carried out through stages.
  - → Firstly, only a number of first stage units are selected.

	→ For each of the selected first stage sampling units, a number of second stage sampling units are selected. → The process is carried on until we select the ultimate sampling units. ▪ The coverage is very large. ▪ It also saves computational labour and is cost effective. ▪ It also adds flexibility in sampling process which is lacking in other sampling schemes. ▪ It is less accurate than stratified sampling.
Non-Probability Sampling	No Probability is attached to the member of the population and as such it is based entirely on the judgment of the sampler. ***Purposive / Judgment Sampling*** ▪ It is dependent solely on the discretion of the sampler & he applies his own judgment based on his beliefs, prejudice, whims, and interest to select the sample. ▪ It is purely subjective & no statistical hypothesis can be tested on the basis of this.
Mixed Sampling	It is based partly on some probabilistic law & partly on some pre-decided rule. ***Systematic Sampling*** ▪ It refers to a sampling scheme where the sampling units are selected at regular interval after selecting the first unit at random (with equal probability). ▪ **Linear Systematic Sampling :** If N is a multiple of 'n', then N = nk, where $0 < k < n$, then we are selecting first unit at random from the first k units & thereby selecting every k^{th} unit till the complete, adequate & updated sampling frame comprising all the members of the population is exhausted. ▪ k = **Sample Interval**. ▪ **Circular Systematic Sampling :** If N is not a multiple of 'n', then N = nk + p, p<k & then we select the first unit from first k units at random & thereafter selecting every k^{th} unit in a cyclic order. ▪ It is a very convenient method where a complete & updated Sampling Frame is available. ▪ It is less time consuming, less expensive & simple as compared to other methods of sampling. ▪ If there is an unknown & undetected periodicity in the sampling frame & sampling interval is a multiple of that period, then we are going to get a most biased sample. ▪ Since, it is not Probability Sampling, no statistical inference can be drawn about population parameter.

CONCEPT 04 : SAMPLING FLUCTUATION & SAMPLING DISTRIBUTION

Sampling Fluctuation	The variation in value of a statistic computed from different samples.
Sampling Distribution	▪ The Probability Distribution of a given statistic is known as **Sampling Distribution**. ▪ The Mean of a Statistic, as obtained from its Sampling Distribution, is known as **Expectation**. ▪ The Standard Deviation of the Statistic is known as the **Standard Error (SE)**. ▪ Standard Error (SE) can be regarded as a measure of precision achieved by sampling. ▪ Standard Error (SE) is inversely proportional to the $\sqrt{\text{Sample Size (n)}}$. ▪ Starting with a population of N units, we can draw many a sample of a fixed size 'n'. **(a) Total No. of Samples (With Replacement) =** N^n **(b) Total No. of Samples (Without Replacement) =** ${}^NC_n$

14A. MEASURES OF CENTRAL TENDENCY

CONCEPT 01 : BASICS OF MEASURES OF CENTRAL TENDENCY

Meaning

- It is the tendency of a given set of observations to cluster around a single central (middle) value.
- The single value that represents the given set of observations is described as Measure of Central Tendency.

Different Measures

MEAN	PARTITION VALUES	MODE
(a) Arithmetic Mean (AM) (b) Geometric Mean (GM) (c) Harmonic Mean (HM)	(a) Median (b) Quartiles (c) Deciles (d) Percentiles	

CONCEPT 02 : MEAN

	Discrete Observations	Simple Frequency Distribution	Grouped Frequency Distribution
Arithmetic Mean ($\bar{x}$)	$\frac{\sum x_i}{n}$	$\frac{\sum f_i x_i}{N}$ where, $N = \sum f_i$	$\frac{\sum f_i x_i}{N}$ where, x_i = Mid-Point of Class Interval & $N = \sum f_i$

Author's Note : Learners can ignore Assumed Mean Method in case of Grouped Distribution.

Properties of AM

(a) If all the observations are constant (say, 'k'), then AM = constant (k).

(b) The sum of deviations of a set of observations from their AM is '0' for unclassified data i.e. $\sum(x_i - \bar{x}) = 0$ (for unclassified data) & $\sum f_i(x_i - \bar{x}) = 0$ (for grouped data).

(c) AM is affected by both Change in Origin & Change in Scale i.e. *If $y = a + bx$, then $\bar{y} = a + b\bar{x}$.*

(d) **Combined AM**

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

Geometric Mean (GM)

Discrete Observations	Simple / Grouped Frequency Distribution
$(x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_n)^{1/n}$	$\left(x_1^{f_1} \cdot x_2^{f_2} \cdot x_3^{f_3} \cdot \ldots \cdot x_n^{f_n}\right)^{1/N}$ where, $N = \sum f_i$

Properties of GM

(a) If all the observations are constant (say, 'k'), then GM = constant (k).

(b) GM of xy = GM of x · GM of y

(c) GM of $\frac{x}{y} = \frac{\text{GM of } x}{\text{GM of } y}$

(d) $\log G = \frac{1}{n}\sum \log x$

Harmonic Mean (HM)

Discrete Observations	Simple / Grouped Frequency Distribution
$\frac{n}{\sum 1/x_i}$	$\frac{N}{\sum f_i/x_i}$ where, $N = \sum f_i$

Properties of HM

(a) If all the observations are constant (say, 'k'), then HM = constant (k).

(b) HM of $1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n} = \frac{2}{n+1}$

(c) HM of 2 numbers $= \frac{2xy}{x+y}$

(d) To calculate Average Speed, calculate Harmonic Mean (HM).

(e) **Combined HM**

$$\frac{n_1 + n_2}{\frac{n_1}{H_1} + \frac{n_2}{H_2}}$$

Author's Note : If the question provides weight instead of frequency, replace frequency with weights in formula.

CONCEPT 03 : PARTITION VALUES / FRACTILES

Meaning

- The values dividing a given set of observations into a no. of equal parts are called Partition Values.
- Median is the middle-most value when the observations are arranged either in ascending or descending order of magnitude. It is also known as Positional Average.

Types

	No. of Equal Parts	No. of Partition Values	Symbol
Median	2	1	Me ($\bar{\mu}$)
Quartile	4	3	Q_1, Q_2, Q_3
Decile	10	9	$D_1, D_2, D_3, \ldots, D_9$
Percentile	100	99	$P_1, P_2, P_3, \ldots, P_{99}$

Median (M)

	Discrete Observations	Simple Frequency Distribution	Grouped Frequency Distribution
Median (M)	*Note : In all three cases, series has to be arranged in ascending / descending order of magnitude.*		
	If n is Odd $\left(\frac{n+1}{2}\right)^{th}$ observation **If n is Even** $\frac{\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}}{2}$ observation	1) Calculate Cumulative Frequency (CF). 2) Calculate $\frac{N+1}{2}$ 3) Check C.F. which is just greater than $\frac{N+1}{2}$. 4) Median = Value of 'x' corresponding to (3).	1) Make sure the series is Exclusive. 2) Calculate $N = \sum f_i$. 3) Calculate N/2. 4) Find CF just greater than N/2 & its corresponding class is Median Class (MC) 5) Apply formula : $l + \left[\frac{\frac{N}{2} - CF}{f} \cdot h\right]$ where, ▪ l = Lower Limit of MC ▪ h = Width of MC ▪ f = Frequency of MC ▪ CF = CF of Class Preceding MC

Properties of Median

(a) The sum of absolute deviations ($\sum |x_i - A|$) is minimum when the deviations are taken from Median.

(b) AM is affected by both Change in Origin & Change in Scale.
If $y = a + bx$, then $y_{me} = a + bx_{me}$.

Other Partition Values

Discrete Observations	Simple / Grouped Frequency Distribution
$(n+1)p^{th}$ value	1) Calculate $N = \sum f_i$. 2) Apply formula : $l + \left[\frac{pN - CF}{f} \cdot h\right]$

where,

- $p = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$ for Q_1, Q_2, Q_3
- $p = \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \ldots, \frac{9}{10}$ for $D_1, D_2, D_3, \ldots, D_9$
- $p = \frac{1}{100}, \frac{2}{100}, \frac{3}{100}, \ldots, \frac{99}{100}$ for $P_1, P_2, P_3, \ldots, P_{99}$

CONCEPT 04 : MODE

	Discrete Observations	Simple Frequency Distribution	Grouped Frequency Distribution
Mode	**By Observation :** The value occurring maximum no. of times.	**By Observation :** 'x' corresponding to the highest frequency.	1) Make sure the series is Exclusive. 2) The Class Interval with highest frequency is Modal Class (MC). 3) Apply formula : $l + \left[\frac{f_1 - f_0}{2f_1 - f_0 - f_2} \cdot h\right]$ where, ▪ l = Lower Limit of MC ▪ h = Size of Class Interval ▪ f_0 = Frequency of class preceding MC ▪ f_1 = Frequency of MC ▪ f_2 = Frequency of class succeeding MC
	Properties of Mode (a) Mode is affected by both Change in Origin & Change in Scale. *If* $y = a + bx$, *then* $y_{mo} = a + bx_{mo}$. (b) If all observations have equal frequency, then the distribution has **No Mode**. (c) If multiple observations have maximum frequency then it is **Multi-Modal Distribution**. (If 2 modes, it is known as **Bi-Modal Distribution**)		

CONCEPT 05 : SOME IMPORTANT RESULTS

Result 01	**Relationship between AM, GM & HM** (a) All the observations are distinct (b) All the observations are same (c) Nothing is mentioned in Question (d) For two positive observations 'a' & 'b'	AM > GM > HM AM = GM = HM AM ≥ GM ≥ HM GM^2 = AM x HM
Result 02	**Relationship between Mean, Median & Mode** (a) For Symmetric Data (b) For Moderately Skewed Data [Empirical Relationship]	(a) Mean = Median = Mode (b) Mode = 3 Median–2 Mean; Mean–Mode = 3(Mean – Median)
Result 03	**Best Measure** (a) Overall (b) Open End Classification	Arithmetic Mean Median
Result 04	Which measure is based on all observations?	AM, GM, HM
Result 05	Which measure is based on 50% values?	Median
Result 06	Which measure is least affected by Extreme Values & Sampling Fluctuations?	Median
Result 07	Which measure is Rigidly Defined & easy to comprehend?	AM, GM, HM, Median
Result 08	Which measure is not based on all observations & has no Mathematical Property?	Mode

14B. MEASURES OF DISPERSION

CONCEPT 01 : BASICS OF DISPERSION

Meaning	Dispersion of a given set of observations is defined as the amount of deviation of the observations, usually, from appropriate Measure of Central Tendency.	
Types of Measures of Dispersions	**ABSOLUTE MEASURES**	**RELATIVE MEASURES**
	▪ They is dependent on the unit of the variable. ▪ They are not useful for comparing two or more distributions. ▪ It includes : (a) Range (b) Mean Deviation (c) Standard Deviation (d) Quartile Deviation	▪ They is independent of the unit. ▪ They are useful for comparing two or more distributions. ▪ It includes : (a) Coefficient of Range (b) Coefficient of Mean Deviation (c) Coefficient of Variation (d) Coefficient of Quartile Deviation

CONCEPT 02 : RANGE & COEFFICIENT OF RANGE

	RANGE	COEFFICIENT OF RANGE
Unclassified Data	L – S where, L = Largest Observation; & S = Smallest Observation	$\frac{L - S}{L + S} \times 100$
Grouped Frequency Distribution	UCB – LCB where, UCB = Uppermost Class Boundary; & LCB = Lowermost Class Boundary ***Note : Make sure the data is Exclusive.***	$\frac{UCB - LCB}{UCB + LCB} \times 100$

CONCEPT 03 : MEAN DEVIATION & COEFFICIENT OF MEAN DEVIATION

	MEAN DEVIATION	COEFFICIENT OF MEAN DEVIATION
Meaning	▪ It is defined as the AM of absolute deviations of the observations from an appropriate Measure of Central Tendency. ▪ It takes its Minimum Value when deviations are taken from Median.	-
Unclassified Data	$\frac{1}{n}\sum \lvert x_i - A \rvert$ where, A is taken as Mean / Median accordingly	$\frac{\text{Mean Deviation about A}}{A} \times 100$
Grouped Frequency Distribution	$\frac{1}{N}\sum f_i \lvert x_i - A \rvert$ where, $N = \sum f_i$	

CONCEPT 04 : STANDARD DEVIATION & COEFFICIENT OF VARIATION

	STANDARD DEVIATION (σ)	COEFFICIENT OF VARIATION
Meaning	▪ It is defined as the root mean square deviation when deviations are taken from AM. ▪ It is denoted as S.D. or σ. ▪ The square of SD is known as Variance (σ^2).	-
Unclassified Data	$\sqrt{\frac{\sum (x_i - \bar{x})^2}{n}}$ OR $\sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2}$	$\frac{SD}{AM} \times 100$
Grouped Frequency Distribution	$\sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{N}}$ OR $\sqrt{\frac{\sum f_i x_i^2}{N} - \bar{x}^2}$	

Important Results	▪ SD of any two numbers = $\frac{\text{Range}}{2}$ ▪ SD of first 'n' Natural Numbers = $\sqrt{\frac{n^2 - 1}{12}}$ ▪ **Combined SD** $\sqrt{\frac{n_1 s_1^2 + n_2 s_2^2 + n_1 d_1^2 + n_2 d_2^2}{n_1 + n_2}}$	

CONCEPT 05 : QUARTILE DEVIATION & COEFFICIENT OF QUARTILE DEVIATION

	QUARTILE DEVIATION	COEFFICIENT OF QUARTILE DEVIATION
Formula	$\frac{Q_3 - Q_1}{2}$ Inter Quartile Range = $Q_3 - Q_1$ Hence, Q_D = Semi-Inter Quartile Range	$\frac{Q_3 - Q_1}{Q_3 + Q_1} \times 100$ **OR** $\frac{Q_D}{\text{Median}} \times 100$

CONCEPT 06 : SOME IMPORTANT RESULTS

Result 01	If all the observations are constant, then Range = MD = SD = QD =	**0**
Result 02	Effect of Change of Origin on Range, MD, SD & QD	**No Effect**
Result 03	Effect of Change of Scale on Range, MD, SD & QD *If for any two constants a & b, two variables x and y are given by y = a + bx, then* ▪ $R_y = \|b\|R_x$ ▪ $\sigma_y = \|b\|\sigma_x$ ▪ $MD_y = \|b\|MD_x$ ▪ $QD_y = \|b\|QD_x$	**In the Same Ratio**
Result 04	Relationship between SD, MD & QD ***Note :*** *SD:MD:QD = 15:12:10*	**4SD = 5MD = 6QD**
Result 05	**Best Measures** (a) Overall (b) For Open End Class	 **SD** **QD**
Result 06	Which measure is quickest to compute?	**Range**
Result 07	Which measure is not Based on all observations?	**Range**
Result 08	Which measure is difficult to comprehend & less mathematical?	**MD**
Result 09	Which measure is easy to comprehend & rigidly defined?	**MD, SD & QD**
Result 10	Which measure is less affected by Extreme Observations & Sampling Fluctuations?	**QD**

15. PROBABILITY

CONCEPT 01 : INTRODUCTION TO PROBABILITY

Meaning	▪ Probability means Possibility. ▪ It is a branch of mathematics that deals with the occurrence of a random event.	
Divisions of Probability	**SUBJECTIVE PROBABILITY**	**OBJECTIVE PROBABILITY**
	▪ It is dependent on personal judgment and experience. ▪ It is influenced by the personal belief, attitude & bias of the person applying it.	▪ It is based on data, calculations & logical deductions. ▪ It is independent of personal opinions & beliefs.

CONCEPT 02 : RANDOM EXPERIMENT

Experiment	It is described as a performance that produces certain results.
Random Experiment	An experiment is defined to be random if the results of the experiment depend on chance only i.e. cannot be predicted in advance. *E.g. : Tossing a coin, Rolling a dice, Drawing items from a box containing both defective & non-defective items, Drawings cards from a pack of well-shuffled 52 cards etc.*
Sample Space	The set containing all elementary events of a random experiment. It is denoted by S or Ω.
Events	▪ The results / outcomes of a random experiment are known as events. ▪ It can be defined as the non-empty subset of S. ▪ **Types of Events**

SIMPLE / ELEMENTARY EVENTS	COMPOSITE / COMPOUND EVENT
It cannot be decomposed into further events.	It is one which can be decomposed into two or more events.
E.g. Tossing a coin gives us two simple events, H & T.	*E.g. Tossing a coin twice can be split into events HT & TH.*

CONCEPT 03 : CLASSICAL DEFINITION OF PROBABILITY [PRIORI DEFINITION]

Formula

▪ This is based on Event Based.
▪ This is given by Bernoulli & Laplace.

$$P(A) = \frac{\text{No. of Favourable Outcomes}}{\text{Total No. of Possible Outcomes}}$$

Note : *If we consider only mutually exclusive, exhaustive & equally likely events, then*

$$P(A) = \frac{\text{No. of Mutually Exclusive, Exhaustive \& Equally Likely Events Favourable to A}}{\text{Total No. of Mutually Exclusive, Exhaustive \& Equally Likely Events}}$$

Imp. Results

(a) **The probability of an event lies between 0 & 1 i.e. $0 \le P(A) \le 1$.**
- If Probability of occurrence of an event is 0, it is known as **Impossible Event**.
  E.g. Getting number 7 on a single roll of dice.
- If Probability of occurrence of an event is 1, it is known as **Sure Event**.
  E.g. Getting a number < 7 on a single roll of dice.

(b)

Odds in Favour of Event A	Odds Against Event A
$\frac{\text{No. of Favourable Events to A}}{\text{No. of Unfavourable Events to A}}$	$\frac{\text{No. of Unfavourable Events to A}}{\text{No. of Favourable Events to A}}$

Note : Probability of an Event A = $\frac{\text{No. of Unfavourable Events to A}}{\text{No. of Favourable Events + No. of Unfavourable Events}}$

Limitations

1. It is only applicable when the total no. of events is Finite.
2. It can be used only when the events are Equally Likely / Equi-Probable.
3. In the field of uncertainty or where no prior knowledge is provided, this definition is inapplicable. Hence, it has only a limited field of application like coin tossing, dice throwing, drawing cards etc.

CONCEPT 04 : MUTUALLY EXCLUSIVE / INCOMPATIBLE EVENTS

Meaning

▪ A set of events are known to be Mutually Exclusive, if their simultaneous occurrence is not possible. It means occurrence of one event implies non-occurrence of other events ($A \cap B = \Phi$).

Example	*Once a coin is tossed, we get two mutually exclusive events Heads & Tails.*
CONCEPT 05 : EXHAUSTIVE EVENTS	
Meaning	▪ A set of events are known to be Exhaustive, if one of them must necessarily occur. ▪ $A \cup B = S$
Example	*Once a coin is tossed, the two events Heads & Tails are Exhaustive as no other event can occur.*
Imp. Results	1. Two events are Exhaustive if, $P(A \cup B) = 1$. 2. Three events are Exhaustive if, $P(A \cup B \cup C) = 1$.
CONCEPT 06 : EQUALLY LIKELY / MUTUALLY SYMMETRIC / EQUI-PROBABLE EVENTS	
Meaning	A set of events are known to be Equally Likely / Mutually Symmetric / Equi-Probable when all of them have same probability of occurrence.
Example	*Once a fair coin is tossed, the two events Heads & Tails are Equally Likely.*
Imp. Results	Three Events are Equally Likely if $P(A) = P(B) = P(C)$. ***Note :*** *If the Events A, B & C are Mutually Exclusive & Exhaustive Events, then* $P(A) = P(B) = P(C) = 1$.
CONCEPT 07 : COMPLIMENTARY EVENT	
Meaning	▪ Probability of Non-Occurrence of an Event A is called Complimentary Event of A. ▪ It is denoted by $P(A')$ or $P(A^C)$ or $P(\bar{A})$.
Imp. Results	1. $P(A) + P(A') = 1$. 2. **<u>Probability that only Event A occurs</u>** $P(A - B) = P(A \cap B') = P(A) - P(A \cap B)$ 3. **<u>Probability that only Event B occurs</u>** $P(B - A) = P(B \cap A') = P(B) - P(A \cap B)$
CONCEPT 08 : AXIOMATIC (MODERN) DEFINITION OF PROBABILITY	
Definition	A real valued function P defined on S is known as a probability measure & P(A) is defined as the probability of A, if P satisfies the three axioms.
Axioms	1. $P(A) \geq 0$ for every $A \subseteq S$ 2. $P(S) = 1$ 3. For Mutually Exclusive Events, $P(A \cup B \cup C \ldots.) = P(A) + P(B) + P(C) \ldots.$
CONCEPT 09 : THEOREMS ON TOTAL PROBABILITY [ADDITION THEOREMS]	
Theorem 01	▪ For any two Events A & B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. ▪ In case A & B are Mutually Exclusive Events, $P(A + B) = P(A \cup B) = P(A) + P(B)$. ***Note :*** *This can be generalised as,* $P(A \cup B \cup C \ldots.) = P(A) + P(B) + P(C) \ldots.$
Theorem 02	For any three events A, B & C, $P(A \cup B \cup C \ldots.) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) - P(A \cap B \cap C)$
CONCEPT 10 : CONDITIONAL PROBABILITY AND COMPOUND / JOINT PROBABILITY	
Meaning	The probability of occurrence of two events simultaneously is known as Compound / Joint Probability. It is denoted by $P(A \cap B)$.
Dependent Events (Conditional Probability)	▪ If the occurrence of one event, say B, is **influenced** by the occurrence of another event A. ▪ It is denoted as P(B/A). ▪ $P(B/A) = \frac{P(A \cap B)}{P(A)}$ ▪ $P(A \cap B) = P(A) \cdot P(B/A)$ ▪ $P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B)$
Independent Event	▪ If the occurrence of one event, say B, is **not influenced** by the occurrence of another event A. ▪ It is known as Independent if the following conditions hold true : (i) $P(A \cap B) = P(A) \cdot P(B)$ (iii) $P(B \cap C) = P(B) \cdot P(C)$ (ii) $P(A \cap C) = P(A) \cdot P(C)$ (iv) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

- P(B/A) = P(B)
- If A & B are independent, then the following pairs of events are also independent :
  - (a) A and B'
  - (b) A' and B
  - (c) A' and B'

CONCEPT 11 : RANDOM / STOCHASTIC VARIABLE

Meaning

- It is a function defined on a Sample Space associated with a random experiment assuming any value from R and assigning a real number to each & every sample point of the random experiment.
- It is always denoted by a Capital Letter.
- If it is defined on a Discrete Sample Space, it is known as Discrete Random Variable & can assume either finite or countably infinite number of values.
- If it is defined on a Continuous Sample Space, it is known as Continuous Random Variable & can assume uncountably infinite number of values.

Example

If a coin is tossed three times & if X denotes the number of heads, then X is a random variable. The Sample Space is given by S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} and X = {0, 1, 2, 3}.

Probability Distribution

From above example, the Probability Distribution of X would look like :

X	P
0	1/8
1	3/8
2	3/8
3	1/8
	1

IMPORTANT RESULTS

(i) $p_i \geq 0$ for every i.

(ii) $\sum p_i = 1$ (over all i)

Expected Value

EXPECTED VALUE	**E(x)**	$\sum p_i x_i$
VARIANCE	**V(x) or σ^2**	$E(x - \mu)^2$ or $E(x^2) - \mu^2$
STANDARD DEVIATION	**σ**	$\sqrt{V(x)}$

Note : If y = a + bx, then the

- Mean (Expected Value) of y is given by $\mu_y = a + b\mu_x$; and
- Standard Deviation of y is given by $\sigma_y = |b|\sigma_x$

Properties of Expected Value

(a) E(k) = k, for any constant k.

(b) E(x + y) = E(x) + E(y) for any two random variables x and y.

(c) E(kx) = k·E(x) for any constant k.

(d) E(xy) = E(x) · E(y) for any two random variables x and y.

Probability as a Function of X

DISCRETE RANDOM VARIABLE	CONTINUOUS RANDOM VARIABLE
Probability Mass Function	Probability Density Function
Conditions (a) $f(X) \geq 0$ for every X. (b) $\sum f(X) = 1$ (over all i) where, f(X) = P(X = x)	**Conditions** (a) $f(X) \geq 0$ for every $x \in [\alpha, \beta]$. (b) $\int_{\alpha}^{\beta} f(x)\,dx = 1$ (over all i) where, f(X) = P(X = x)
Expected Value (μ) = $\sum xf(x)$	**Expected Value (μ) =** $\int_{-\infty}^{\infty} xf(x)\,dx$
Variance (σ^2) = $E(x^2) - \mu^2$	**Variance (σ^2) =** $E(x^2) - \mu^2$ where, $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

***Note :** The probability that x lies between two specified values a and b, where $\alpha \leq a < b \leq \beta$, is given by* $\int_a^b f(x)\,dx$.

16. THEORETICAL DISTRIBUTIONS

CONCEPT 01 : BASICS OF THEORETICAL DISTRIBUTION

Need of Theoretical Distributions

- In our previous chapter, we answered questions like, "If I draw one card from a shuffled deck, what is the probability it is a King?" or "If I roll two dice, what is the probability the sum is 7?"
- Our approach was largely *empirical* or *classical*; we counted favourable outcomes and divided by the total number of possible outcomes.

Now, we ask a more profound question:
Can we create a universal mathematical model that describes the probabilistic behaviour of a system for all its possible outcomes?

Types of Theoretical Distributions

DISCRETE PROBABILITY DISTRIBUTIONS	CONTINUOUS PROBABILITY DISTRIBUTIONS
▪ Binomial Distribution ▪ Poisson Distribution	Normal Distribution

CONCEPT 02 : DISCRETE PROBABILITY DISTRIBUTIONS

	BINOMIAL DISTRIBUTION	POISSON DISTRIBUTION
Introduction	It is derived from Bernoulli's Trials. **Characteristics of the Trials** (a) Mutually Exclusive & Exhaustive Outcomes (b) Independent of Each Other (c) Probability of Success (p) & Failure (q = 1 – p), remain unchanged. (d) No. of trials is a finite positive integer.	It applies when the total number of events is very large but the occurrence is very small. **Characteristics** (a) Probability of finding success in a very small time interval (t, t+dt) is kt, where k (>0) is a constant. (b) Probability of >1 success is very low. (c) Probability of having success is independent of 't' & earlier successes.
Notation	A discrete random variable 'x' is defined to follow Binomial Distribution with parameters 'n' & 'p', **x ~ B (n,p)** ***Bi-Parametric Discrete Prob. Distribution***	A discrete random variable 'x' is defined to follow Poisson Distribution is denoted as, **x ~ P(m)**, where m = np ***Uni-Parametric Discrete Prob. Distribution***
Probability Mass Function	$f(x) = p(X = x)$ $= {}^nC_x \cdot p^x q^{n-x}$, **for $x \in W$** $= 0$, **otherwise**	$f(x) = p(X = x)$ $= \dfrac{e^{-m} \cdot m^x}{x!}$ **for $x \in W$** $= 0$, **otherwise**
Mean (μ)	np	m
Variance (σ^2)	npq **OTHER POINTS** ▪ Variance of Binomial Variable is always less than its mean. ▪ It obtains its maximum value $\frac{n}{4}$ at p=q=0.5. *{It is symmetric if p=q=0.5 only}*	m ***Poisson's Approximation to Binomial*** *If 'n' tends to infinity & 'p' tends to 0, then m = np is finite, and B(n,p)* ≅ P(m).
Mode (μ_0)	(n + 1)p	m

	Binomial: It is an Integer	Binomial: It is not an Integer	Poisson: It is an Integer	Poisson: It is not an Integer
Mode (μ_0)	**Bi-Modal** (n+1)p & (n+1)p - 1	**Uni-Modal** (n+1)p (Rounded Down)	**Bi-Modal** m & m - 1	**Uni-Modal** m (Rounded Down)

	BINOMIAL DISTRIBUTION	POISSON DISTRIBUTION
Additive Property	If x & y are two independent variables such that $x \sim B(n_1, p)$ & $y \sim B(n_2, p)$, then $x+y \sim B(n_1+n_2, p)$.	If x & y are two independent variables such that $x \sim P(m_1)$ & $y \sim P(m_2)$, then $x+y \sim P(m_1+m_2)$.
Applications	▪ Coin Tossing Experiments ▪ Sampling Inspection Plan ▪ Genetic Experiments etc.	▪ Printing Mistakes per page ▪ Road accidents per minute ▪ Radio-active elements per minute ▪ No. of demands per minute

CONCEPT 03 : NORMAL / GAUSSIAN DISTRIBUTION

Meaning

- It is the most widely known & used of all distributions.
- In case of a continuous random variable like height or weight, it is impossible to distribute the total probability among different mass points because between any two unequal values, there remains an infinite no. of values.

Notation

A continuous random variable 'x' is defined to follow Normal Distribution with parameters 'μ' & 'σ^2' to be denoted by,

$$x \sim N(\mu, \sigma^2)$$

Bi-Parametric Continuous Prob. Distribution

Probability Density Function

$$f(x) = \frac{1}{\mu\sqrt{2\pi}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

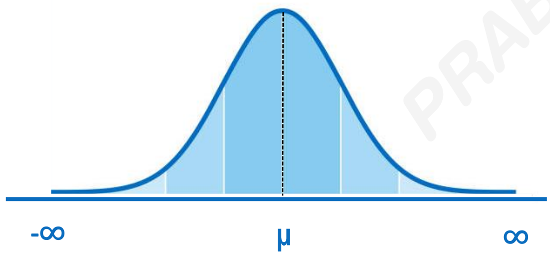
where,

- e = 2.71828
- x = A random variable, $-\infty < x < \infty$
- μ = Mean of x
- σ = Standard Deviation of x

Properties of Normal Curve

- Bell Shaped Curve (aka Probability Curve)
- The line drawn through x = μ divides the curve into two equal parts.
- The curve is Symmetric (i.e. Skewness = 0) at x = μ.
- The two tales of the Normal Curve extends indefinitely on both sides & never touch the horizontal axis.
- Total Area of Normal Curve = 1 (Unity), hence,
  - (i) Area from $-\infty$ to μ = 0.5
  - (ii) Area from μ to ∞ = 0.5

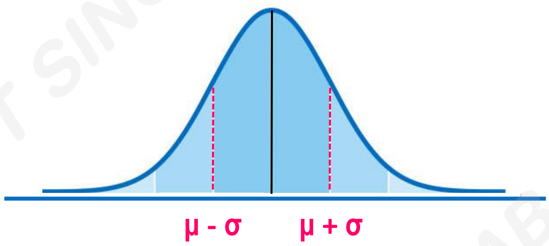
Normal (z) Curve



Statistical Results

- Mean = Median = Mode = μ {Symmetric Distribution}
- **<u>Variance (σ^2) is generally given in question</u>**
  - → Standard Deviation = σ
  - → Mean Deviation = 0.8σ
  - → Quartile Deviation = 0.675σ
  - → Quartiles
    - (i) Q_1 : μ - 0.675σ
    - (ii) Q_3 : μ + 0.675σ

Points of Inflexion



Points of Inflexion

Sums of Independent Normal Variables

If x & y are independent normal variables with Means & Standard Deviations as μ_1, μ_2 & σ_1, σ_2 respectively, then z = x + y also follow Normal Distribution with,

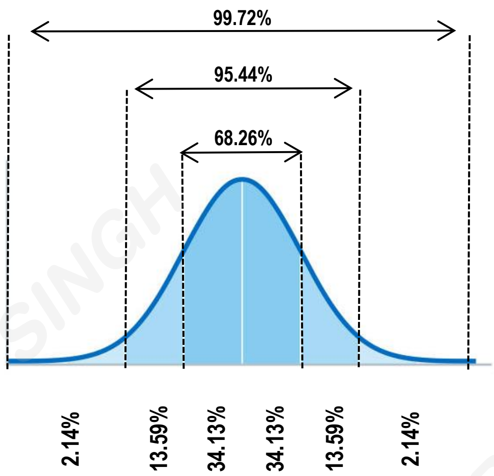
- Mean = $\mu_1 + \mu_2$
- S.D. = $\sqrt{\sigma_1^2 + \sigma_2^2}$

Applications

These approach Normal Distribution in certain cases :

(a) 'n' is Large & 'p' is moderate in Binomial.
(b) 'm' is Large in Poisson Distribution.
(c) Probability Distributions of t, chi-square and F also tends to Normal Distribution.
(d) Sample Statistic Approach for large sample.

Area under Normal Curve



CONCEPT 04 : STANDARD NORMAL DISTRIBUTION

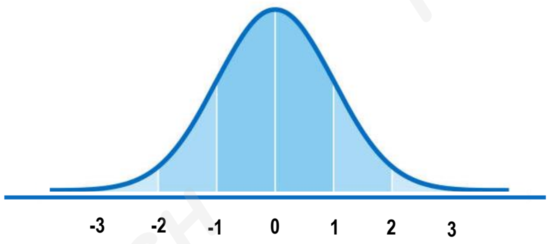
Meaning

If $\mu = 0$ & $\sigma = 1$,

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-z^2}{2}}, \text{ for } -\infty < z < \infty$$

$$z = \frac{x - \mu}{\sigma}$$

The random variable ‘z’ is known as **Standard Normal Variate / Deviate.**



Statistical Results

- Mean = Median = Mode = 0 {Symmetric Distribution}
- Variance = SD = 1
- Points of Inflextion = -1 & 1
- Mean Deviation = 0.8
- Quartile Deviation = 0.675

Author’s Note :

- **Φ(n) means -∞ to n.**
- **z = n means 0 to n.**

17. CORRELATION & REGRESSION

CONCEPT 01 : BIVARIATE DATA

Meaning
- When data are collected on two variables simultaneously, they are known as Bivariate Data.
- The frequency distribution derived is known as Bivariate Frequency Distribution.

Bivariate Frequency Distribution

		VARIABLE A					
	X \ Y	a - b	c - d	e - f	g - h	i - j	Total
VARIABLE B	a - b						
	c - d						
	e - f						
	Total						

IMPORTANT RESULTS

(a) **Total No. of Cells =** m (Class Intervals of X) · n (Class Intervals of Y)

(b) **No. of Marginal Distributions =** 2

(c) **No. of Conditional Distributions =** m + n

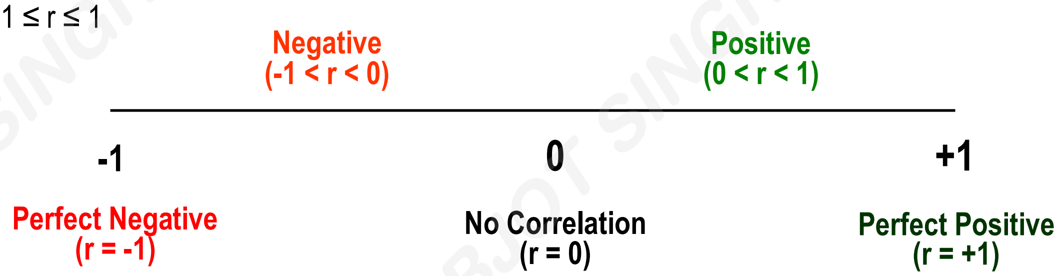
CONCEPT 02 : CORRELATION ANALYSIS

Meaning

Is change in one variable reciprocated by a corresponding change in other variable?

YES		NO
Associated / Correlated		Disassociated / Uncorrelated / Independent *E.g. Shoe-Size & Intelligence*
Is the Direction of Change same?		
YES	NO	
Positive Correlation *E.g. Yield & Rainfall*	Negative Correlation *E.g. Price & Demand*	

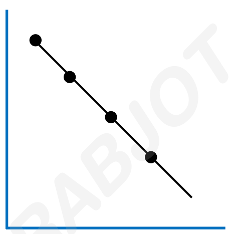
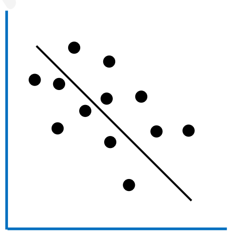
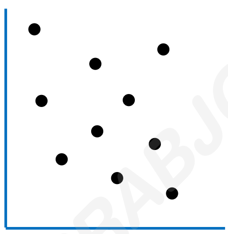
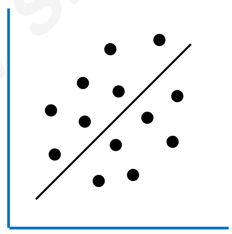
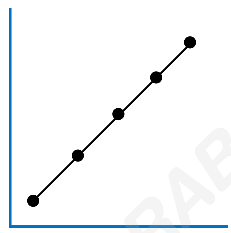
Notation
- It is denoted by 'r'.
- $-1 \le r \le 1$



Measures of Correlation
1. Scatter Diagram
2. Karl Pearson's Product Moment Correlation Coefficient
3. Spearman's Rank Correlation Coefficient
4. Coefficient of Concurrent Deviations

CONCEPT 03 : SCATTER DIAGRAM

Meaning
- It is a simple diagrammatic method to establish correlation between a pair of variables.
- It can be applied for any type of correlation, linear as well as curvilinear.
- It can distinguish between different types of correlation.
- It fails to measure the extent of relationship between the variables.

Perfect Negative (r = -1)	Negative (-1 < r < 0)	No Correlation (r = 0)	Positive (0 < r < 1)	Perfect Positive (r = +1)
				

CONCEPT 04 : KARL PEARSON'S PRODUCT MOMENT CORRELATION COEFFICIENT

Meaning

- It can be defined as the ratio of covariance between the two variables to the product of standard deviations of the two variables.
- It is the best method for finding correlation between two variables provided the relationship between them is Linear.

Computation

$$\frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{n} = \frac{\sum x_i y_i}{n} - \bar{x}\bar{y}$$

$$r = r_{xy} = \frac{Cov(x, y)}{S_x \cdot S_y}$$

$$\sqrt{\frac{\sum(x_i - \bar{x})^2}{n}} = \sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2} \qquad \sqrt{\frac{\sum(y_i - \bar{y})^2}{n}} = \sqrt{\frac{\sum y_i^2}{n} - \bar{y}^2}$$

A single formula can be given as,

$$r = \frac{n\sum x_i y_i - \sum x_i \cdot \sum y_i}{\sqrt{n\sum x_i^2 - (\sum x_i)^2} \cdot \sqrt{n\sum y_i^2 - (\sum y_i)^2}}$$

Properties

(a) The Coefficient of Correlation is a **Unit-Free Measure**.

(b) The Coefficient of Correlation always lies between -1 & 1 i.e. $\mathbf{-1 \le r \le 1}$

(c) If two variables are related by a Linear Equation, then Correlation Coefficient will be :

$y = a + bx$	
If $b > 0$	$r = -1$
If $b < 0$	$r = 1$

Author's Note : In $y = a + bx$, a is known as Intercept & b is known as Slope.

(d) The Coefficient of Correlation is not affected by Change of Origin.

(e) The Coefficient of Correlation is not affected by value of Change of Scale, but is affected by the sign.

- **If sign of Change of Scale in both variables are same :** $r_{uv} = r_{xy}$
- **If sign of Change of Scale in both variables are different :** $r_{uv} = -r_{xy}$

If x & y, two variables, are changed to a pair of new variables u & v, such that, $u = \frac{x - a}{b}$ & $v = \frac{y - c}{d}$, then, $r_{xy} = \frac{bd}{|b||d|} r_{uv}$.

CONCEPT 05 : SPEARMAN'S RANK CORRELATION COEFFICIENT METHOD

Meaning

- When we need to find correlation between two **qualitative characteristics**, say, beauty & intelligence, we use this method.
- It can also be applied to find the level of agreement (/ disagreement) between two judges so far as assessing a qualitative characteristic is concerned.

Computation

$$r_R = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$$

where,

- r_R = Rank Correlation Coefficient ($-1 \le r_R \le 1$)
- $d_i = x_i - y_i$ represents the difference in ranks for the i^{th} individual
- n = No. of Individuals

Tied Rank	In case 'u' individuals receive the same rank, we describe it as a tied rank of length 'u'. $$r_R = 1 - \frac{6\left[\sum d_i^2 + \sum \frac{(t_j^3 - t_j)}{12}\right]}{n(n^2-1)}$$ where, ▪ t_j = j^{th} tie length ▪ $\sum(t_j^3 - t_j)$ extends over the length of all the ties for both the series

CONCEPT 06 : COEFFICIENT OF CONCURRENT DEVIATIONS

Meaning	It is a very simple & casual method of finding correlation when we are not very serious about the magnitude of the two variables.
Computation	**Step 01 :** Attach a positive sign, if the value is more than the previous value & attach a negative sign if the value is less than the previous value. **Step 02 :** The deviation in x value & corresponding y value is known to be concurrent if both deviations have the same sign. $$r_C = \pm\sqrt{\pm\frac{(2c - m)}{m}}$$ where, ▪ c = No. of Concurrent Deviations (Deviations having same sign) ▪ m = No. of Pairs compared (n) - 1

CONCEPT 07 : REGRESSION ANALYSIS

Meaning	In Regression Analysis we are concerned with the estimation of one variable for a given value of another variable(s) on the basis of an average mathematical relationship between the two (or more) variables.	
Simple Linear Regression	**ESTIMATION OF Y WHEN X IS GIVEN**	**ESTIMATION OF X WHEN Y IS GIVEN**
	▪ y = Dependent Variable / Regression / Explained Variable ▪ x = Independent Variable / Predictor / Explanator	▪ x = Dependent Variable / Regression / Explained Variable ▪ y = Independent Variable / Predictor / Explanator
	$y = a + bx$ where, ▪ a & b are constants [Regression Parameters] ▪ b is known as Regression Coefficient of y on x i.e. b_{yx}.	$x = a + by$ where, ▪ a & b are constants [Regression Parameters] ▪ b is known as Regression Coefficient of x on y i.e. b_{xy}.
	Using Method of Least Squares Regression Line of y on x $y - \bar{y} = b_{yx}(x - \bar{x})$	**Using Method of Least Squares Regression Line of x on y** $x - \bar{x} = b_{xy}(y - \bar{y})$
	$b_{yx} = \frac{Cov(x, y)}{Variance\ of\ x} = r \cdot \frac{\sigma_y}{\sigma_x}$	$b_{xy} = \frac{Cov(x, y)}{Variance\ of\ y} = r \cdot \frac{\sigma_x}{\sigma_y}$
Properties	(a) Regression Coefficients remain unchanged due to Change of Origin but change due to a Shift of Scale. $b_{uv} = b_{yx} \cdot \frac{Change\ of\ Scale\ of\ y}{Change\ of\ Scale\ of\ x}$ **&** $b_{uv} = b_{xy} \cdot \frac{Change\ of\ Scale\ of\ x}{Change\ of\ Scale\ of\ y}$ (b) The two lines of regression intersect at the point $(\bar{x}, \bar{y})$ i.e. Mean, which is also the solution of the simultaneous equations in x & y. (c) $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$ [If both are negative, r would be negative & if both are positive, r would be positive] (d) Product of the regression coefficients must be numerically less than 1 (unity).	

	(e) Regression can be applied, unlike Correlation, for both Linear & Curvilinear Relationships. (f) The two lines of Regression coincide (become identical) when r = -1 or +1. (g) The two lines of Regression are perpendicular to each other when r = 0.

CONCEPT 08 : COEFFICIENT OF DETERMINATION / EXPLAINED VARIANCE / ACCOUNTED VARIANCE	
Meaning	Correlation Coefficient measuring a linear relationship between the two variables indicates the amount of variation of one variable accounted for by another variable.
Computation	r^2 ***Note : Coefficient of Non-Determination / Unexplained Variance / Unaccounted Variance is given by $1 - r^2$.***

18. INDEX NUMBERS

CONCEPT 01 : BASICS OF INDEX NUMBERS

Meaning

- It is a ratio of two or more time periods are involved, one of which is the base time period. The value at the base time period serves as the standard point of comparison. *E.g. NSE, BSE etc.*
- They are convenient devices for measuring relative changes of differences from time to time or place to place.
- An **Index Time Series** is a list of index numbers for two or more periods of time, where each index number employs the same base year.
- Index Number for the Base Year is always taken as 100.

Issues Involved

Selection of Data	▪ It is necessary to understand the purpose for which the index is used. ▪ It is necessary to ensure that the sample is representative. ▪ It is necessary to ensure comparability of data.
Base Period	▪ It is a point of reference in comparing various data. ▪ It should be a normal period of relative stability (i.e. not of war, famine etc.) ▪ It should be relatively recent.
Selection of Weights	Each variable involved in composite index should have a reasonable influence on the index.
Use of Averages	Geometric Mean is better at averaging relatives, but for most of the indices Arithmetic Mean is used because of its simplicity.

Relatives

They are derived because absolute numbers measured in some appropriate unit, are often of little importance and meaningless in themselves.

FOR INDIVIDUAL COMMODITY

PRICE RELATIVE	QUANTITY RELATIVE	VALUE RELATIVE
$\frac{P_n}{P_o} \times 100$	$\frac{Q_n}{Q_o} \times 100$	$\frac{V_n}{V_o} \times 100 = \frac{P_nQ_n}{P_oQ_o} \times 100$

Link Relatives

When successive prices / quantities are taken, the relatives are called Link Relative.

$$\frac{P_1}{P_0}, \frac{P_2}{P_1}, \frac{P_3}{P_2}, \frac{P_n}{P_{n-1}}$$

Chain Relatives

When the above relatives are in respect to a fixed base period these are also called :

- Chain Relatives with respect to this base; or
- Relatives chained to a fixed base

$$\frac{P_1}{P_0}, \frac{P_2}{P_0}, \frac{P_3}{P_0}, \frac{P_n}{P_0}$$

Methods

SIMPLE INDEX NUMBER	WEIGHTED INDEX NUMBER
▪ Aggregative Method ▪ Relative Method	▪ Aggregative Method (a) Laspeyres' Index (b) Paasche's Index (c) Marshall Edgeworth Index (d) Fisher's Index ▪ Relative Method

CONCEPT 02 : SIMPLE INDEX NUMBERS

	SIMPLE AGGREGATIVE METHOD	SIMPLE AVERAGE OF RELATIVES
Formula	$\sum \frac{P_n}{P_0} \times 100$ where, ▪ P_n = Current Year Prices ▪ P_0 = Base Year Prices	$\frac{\Sigma\left[\frac{P_n}{P_0} \cdot 100\right]}{N}$
Merits	Easy to understand	They are pure numbers. Hence, Price Index Number computed from relatives will remain the same even if units are changed.

Demerits	▪ Commodities with higher prices exert greater influence. ▪ Index Number changes if units are changed.	It gives equal importance to each of the relatives. This can be remedied by introduction of appropriate weighing system.

CONCEPT 03 : WEIGHTED INDEX NUMBERS

	WEIGHTED AGGREGATIVE INDEX	WEIGHTED AVERAGE OF RELATIVES
Formulae	**Laspeyres' Index** $\sum \frac{P_nQ_0}{P_0Q_0} \times 100$ **Paasche's Index** $\sum \frac{P_nQ_n}{P_0Q_n} \times 100$ **Marshall-Edgeworth Index** $\sum \frac{P_n(Q_0+Q_n)}{P_0(Q_0+Q_n)} \times 100$ **Fisher's Ideal Price Index** GM of Laspeyres' & Paasche's Index. $\sqrt{\sum \frac{P_nQ_0}{P_0Q_0} \cdot \sum \frac{P_nQ_n}{P_0Q_n}} \times 100$ **Bowley's Index** AM of Laspeyres' & Paasche's Index.	$\sum \frac{P_nQ_0}{P_0Q_0} \times 100$ {Same as Laspeyres' Index}
	Note : General Index No. = $\frac{\sum w_iI_i}{\sum w}$ ***, where I = Index Numbers given in question.***	

CONCEPT 04 : CHAIN INDEX NUMBERS

Meaning	▪ So far we concentrated on a fixed base, but it does not suit when conditions change quite fast. ▪ Under this method, the relatives of each year are first related to the preceding year called link relatives & then they are chained together by successive multiplication to form a chain index.
Computation	$\frac{\text{Link Relative of Current Year x Chain Index of Previous Year}}{100}$

CONCEPT 05 : QUANTITY INDEX NUMBERS

Simple Quantity Index Nos.	**SIMPLE AGGREGATE OF QUANTITY** $\sum \frac{Q_n}{Q_0} \times 100$	**SIMPLE AVG. OF QUANTITY RELATIVES** $\frac{\sum \left[\frac{Q_n}{Q_0} \cdot 100\right]}{N}$
Weighted Qty. Index Nos.	**WEIGHTED AGGREGATE QUANTITY** **Laspeyres' Index** $\sum \frac{Q_nP_0}{Q_0P_0} \times 100$ **Paasche's Index** $\sum \frac{Q_nP_n}{Q_0P_n} \times 100$ **Fisher's Ideal Price Index** GM of Laspeyres' & Paasche's Index. $\sqrt{\sum \frac{P_nQ_0}{P_0Q_0} \cdot \sum \frac{P_nQ_n}{P_0Q_n}} \times 100$	**WEIGHTED AVERAGE OF QTY. RELATIVES** $\sum \frac{Q_nP_0}{Q_0P_0} \times 100$ {Same as Laspeyres' Index}

CONCEPT 06 : VALUE INDICES	
Formula	$\sum \frac{V_n}{V_0} = \sum \frac{P_n Q_n}{P_0 Q_0}$

CONCEPT 07 : DEFLATING TIME SERIES USING INDEX NUMBERS	
Formula	▪ Deflated Value = $\frac{\text{Current Value}}{\text{Price Index of Current Year}}$ = Current Value x $\frac{P_0}{P_n}$ ▪ Real Wages = $\frac{\text{Actual Wages}}{\text{Cost of Living Index}}$ x 100

CONCEPT 08 : SHIFTING & SPLICING OF INDEX NUMBERS	
Shifting of Index Numbers	Shifted Price Index = $\frac{\text{Original Price Index}}{\text{Price Index for the year on which it has to be shifted}}$ x 100
Splicing of Index Numbers	Two Index Numbers covering different bases may be combined into a single series by splicing. It is usually required when there is a major change in quantity weights.

CONCEPT 09 : TEST OF ADEQUACY	
Unit Test	▪ This test requires that the formula should be independent of the unit in which or for which prices & quantities are quoted. ▪ All formulae satisfy this test. ***[Exception : Simple (Unweighted) Aggregative Index]***
Time Reversal Test	▪ It is a test to determine whether a given method will work both ways in time, forward & backward. ▪ It provides that the formula for calculating the index number should be such that two ratios, the current on the base & the base on the current should multiply into unity. In other words, the two indices should be reciprocals of each other. $P_{01} \times P_{10} = 1$ ▪ **Laspeyre's & Paasche's Formula doesn't satisfy this test**, but **Fisher's does**.
Factor Reversal Test	▪ This holds when the product of the Price Index & the Quantity Index should be equal to the corresponding Value Index. $P_{01} \times Q_{01} = V_{01}$ ▪ **Fisher's Index satisfies this.** ***Because, Fisher's Index satisfies both Time Reversal & Factor Reversal Tests, it is called, FISHER'S IDEAL INDEX NUMBER***
	Note : While selecting an appropriate index formula, the Time Reversal Test & Factor Reversal Test are considered necessary in testing the consistency.
Circular Test	▪ It is concerned with the measurement of price changes over a period of years, when it is desirable to shift the base. ▪ This property therefore enables us to adjust the index values from period to period without referring each time to the original base. ▪ This test of shiftability of base is called the Circular Test. $P_{01} \times P_{12} \times P_{20} = 1$ ▪ **This test is not met by Laspeyre's, Paasche's or Fisher's Index.** ▪ **Simple GM of Price Relatives & Weighted Aggregative with Fixed Weights meet this test.**