9 LIMITS AND CONTINUITY

LEARNING OBJECTIVES

After studying this chapter, you will be able to:

- Know the concept of limits and continuity;
- Understand the theoruems underlying limits and their applications; and
- Know how to solve the problems relating to limits and continuity with the help of given illustrations.

Properties of Limits:

(i) $\lim_{x\to a}\{f_1(x) \pm f_2(x)\} = \lim_{x\to a} f_1(x) \pm \lim_{x\to a} f_2(x)$

(ii) $\lim_{x\to a}\{f_1(x).f_2(x)\} = \lim_{x\to a} f_1(x).\lim_{x\to a} f_2(x)$

(iii) $\lim_{x\to a}\left[{f_1(x)}/{f_2(x)}\right] = {\lim_{x\to a} f_1(x)}/{\lim_{x\to a} f_2(x)}$

(iv) $\lim_{x\to a}[k.f(x)] = k.\lim_{x\to a} f(x)$ ['k' is a constant term]

(v) $\lim_{x\to a} k = k$ ['k' is a constant term]

Meaningless Form:

$\frac{0}{0}, \infty, 0^0, 0^\infty, \frac{1}{0}$

Some Expansions:

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\cdots\cdots\cdots\cdots$

(ii) $a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + \cdots\cdots\cdots\cdots\cdots\cdots$

(iii) $\log(1+\mathrm{x}) = \mathrm{x} - \frac{\mathrm{x}^2}{2} + \frac{\mathrm{x}^3}{3} - \frac{\mathrm{x}^4}{4} + \cdots\cdots\cdots\cdots\cdots\cdots$

(iv) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\cdots\cdots\cdots\cdots\cdots$

Some Important Limits:

(i) If $f(x)$ is a polynomial, then $\lim_{x\to a} f(x) = f(a)$

(ii) If $a \neq 0$ and $n \in Q$, then $\lim_{x\to a}\frac{x^n - a^n}{x - a} = na^{n-1}$

(iii) $\lim_{x\to a}\frac{x^m - a^m}{x^n - a^n} = \frac{m}{n}.a^{m-n}$

(iv) $\lim_{x\to 0}\frac{e^x - 1}{x} = 1$

(v) $\lim_{x\to 0}\frac{a^x - 1}{x} = \log_e a$

(vi) $\lim_{x\to 0}(1 + x)^{\frac{1}{x}} = e$

(vii) $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$

(viii) $\lim_{x\to 0}\frac{1}{x}$ *does not exits*

(ix) $\lim_{x\to 0}\frac{(1+x)^n - 1}{x} = n$

(x) $\lim_{x\to 0}\frac{(1-x)^n - 1}{x} = -n$

(xi) $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

(xii) $\lim_{x\to 0}\frac{\log(1-x)}{x} = -1$

CONTINUITY:

A function f(x) is said to be continuous if at that point L.H.L (Left Hand Limit) = R.H.L (Right Hand Limit) and if L.H.L is not is equal to R.H.L. Therefore, the function f(x) will be discontinuous. i.e.,

$$(\text{L.H.L}) \lim_{x\to a^-} - f(x) = \lim_{x\to a^+} + f(x)\ (\text{R.H.L})$$

In other words, required condition for continuity left hand limit and right hand limit should be equal.

If we discuss the continuity at $'x \to 2'$, Then all slightly less than values are called L.H.L and slightly greater than values are called R.H.L. For example

$$\begin{bmatrix} x = 2.02 & x = 2 + 0.02 \\ x = 2.01 & x = 2 + 0.01 \end{bmatrix} x = 2 + h\ or, x > 2\ (\text{R.H.L})$$

$$\boxed{x \to 2}$$

$$\begin{bmatrix} x = 1.99 & x = 2 - 0.01 \\ x = 1.98 & x = 2 - 0.02 \end{bmatrix} x = 2 - h\ or, x < 2\ (\text{L.H.L})$$

Properties of Continuous Functions:

(i) The sum difference or product of two continuous functions is a continuous function. This property bolds good for any finite number of function.

(ii) The quotient of two continuous function is a continuous function, provided the denominator is not O (zero).

(iii) If f(x) is continuous at x = c and f(c) ≠ 0, then f(x) is of the same sign as that of f(c) in the neighbourhood at x = c. For example.

When $\text{f}(x) = 2x.\text{f}\left(\frac{\pi}{2}\right) = 1.\text{f}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \text{f}\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. That is f(x) maintains the positive sign in the neighbourhood of $x = \frac{\pi}{2}$.

EXERCISE

Note : Pick up the correct answer from the following :

1 $\lim_{x\to 0}\left(\frac{\sqrt{2-x}-\sqrt{2+x}}{x}\right)=?$

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\sqrt{2}$ (d) $\frac{-1}{\sqrt{2}}$

2. $\lim_{x\to 2}\left(\frac{x^3\sqrt{x}-8\sqrt{2}}{x-2}\right)=?$

(a) $4\sqrt{2}$ (b) $-16\sqrt{2}$ (c) $14\sqrt{2}$ (d) None

3. $\lim_{x\to 2}\left(\frac{x^5-32}{x^3-8}\right)=?$

(a) 4 (b) $\frac{20}{3}$ (c) $\frac{1}{5}$ (d) $\frac{3}{8}$

4. $\lim_{x\to 0}\frac{a^x-1}{b^x-1}=?$

(a) $\log_b a$ (b) $\log_a b$ (c) $\log\left(\frac{a}{b}\right)$ (d) $\log\left(\frac{b}{a}\right)$

5. $\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\cdots\cdots\cdots+\frac{1}{2^n}\right)=?$

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) None

6. $\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^{3x}=?$

(a) e (b) e^3 (c) e^{12} (d) None

7. If f(x) = $\begin{cases} 2x-1 & where\ x<2 \\ \frac{3x}{2} & where\ x\geq 2 \end{cases}$, then $\lim_{x\to 2} f(x)=?$

(a) 3 (b) $\frac{3}{2}$ (c) 2 (d) Does not exist

8. Find the value of k for which the function.

$F(x) = \begin{cases} \frac{x^2 - 2x - 3}{x + 1} \\ k \end{cases}$ is continuous at x = – 1, find k

(a) 4 (b) – 4 (c) 2 (d) 3

9. $\lim_{x \to 0} \left\{ \frac{(x+2)^{5/3} - (a+2)^{5/3}}{x - a} \right\} =?$

(a) $\frac{5}{3} a^{2/3}$ (b) $\frac{2}{3}(a+2)^{2/3}$ (c) $\frac{5}{3}(a+2)^{2/3}$ (d) None of these

10. $\lim_{x \to a} \left(\frac{\sqrt{x} - \sqrt{2}}{\sqrt[3]{x} - \sqrt[3]{2}} \right) = ?$

(a) $\frac{3}{2} 2^{1/6}$ (b) $\frac{2}{3} 2^{1/6}$ (c) $\frac{3}{2^{7/6}}$ (d) None of these

11. $\lim_{x \to a} \left(\frac{\sqrt{a + 2x} - \sqrt{3x}}{\sqrt{3a + x} - 2\sqrt{x}} \right) = ?$

(a) $\frac{2 - \sqrt{3}}{a - 2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{2}{3\sqrt{3}}$ (d) None of these

12. $\lim_{x \to 1} \left(\frac{\log x}{x - 1} \right) = ?$

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) None of these

13. $\lim_{x \to 0} \left(\frac{a^x - b^x}{x} \right) = ?$

(a) 0 (b) 1 (c) $\log \frac{a}{b}$ (d) $\log \frac{b}{a}$

14. $\lim_{x \to 0} \frac{(1 + x)^n - 1}{x} = ?$

(a) 1 (b) n (c) (n – 1) (d) None

15. $\lim_{x \to 0} \left(\frac{e^{x} - e^{-x}}{x} \right) = ?$

(a) 2 (b) $\frac{1}{2}$ (c) 0 (d) 1

16. $\lim_{x \to 0} \left(\frac{(1+x)^6 - 1}{(1+x)^2 - 1} \right) = ?$

(a) 3 (b) 6 (c) 1 (d) 0

17. $\lim_{x \to 0} \frac{\sqrt{x^6 + 16} - 4}{\sqrt{x^2 + 9} - 3} = ?$

(a) $\frac{3}{4}$ (b) $-\frac{3}{4}$ (c) $\frac{4}{3}$ (d) $-\frac{4}{3}$

18. $\lim_{x \to 0} \left(\frac{e^{ax} - e^{bx}}{x} \right) = ?$

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$ (c) (a – b) (d) $\log \frac{a}{b}$

19. $\lim_{x \to 0} \left(\frac{\log(1+x)}{2^x - 1} \right) = ?$

(a) $\log_e 2$ (b) $\log_2 e$ (c) $\frac{1}{2}$ (d) None of these

20. $\lim_{x \to 0} \left(\frac{\sqrt{1+x^n} - \sqrt{1-x^n}}{x^n} \right) = ?$

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) None of these

21. $\lim_{x \to 0} \left(\frac{1 - \sqrt{1 - x^2}}{x^2} \right) = ?$

(a) $\frac{1}{2}$ (b) 2 (c) 1 (d) – 1

22. $\lim_{x \to 1} \left(\frac{1}{\log x} - \frac{x}{\log x} \right) = ?$

(a) – 1 (b) 1 (c) – 2 (d) 2

23. $\lim_{x\to 1}\left\{\frac{2}{x^2-1} - \frac{1}{x-1}\right\} = ?$

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 2 (d) – 2

24. $\lim_{x\to 0}\left(\frac{1}{e^x-1} - \frac{1}{x}\right) = ?$

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) 0

25. $\lim_{x\to 0}\frac{x}{e^x} = ?$

(a) 0 (b) 1 (c) e (d) ∞

26. $\lim_{x\to 2}\frac{2x^2-5x+2}{x^2-3x+2} = ?$

(a) 8 (b) 3 (c) 2 (d) 1

27. $\lim_{x\to 1}\frac{x^5-2x^3-4x^2+9x-4}{x^4-2x^3+2x-1} = ?$

(a) 4 (b) – 2 (c) 9 (d) $\frac{-3}{2}$

28. $\lim_{x\to \infty}\frac{3x^2-5x+6}{x^3+4x^2+x-3} = ?$

(a) 0 (b) 3 (c) – 2 (d) None of these

29. $\lim_{x\to \infty}\frac{5x^3-6x^2+4x-3}{4x^3+5x^2+8x+2} = ?$

(a) 5 (b) $\frac{1}{4}$ (c) $-\frac{3}{2}$ (d) $\frac{5}{4}$

30. $\lim_{x\to \infty}\frac{1}{(1-x)^2} = ?$

(a) 0 (b) 1 (c) ∞ (d) None of these

31. $\lim_{x\to\infty} \frac{4x^3 - 2x^2 + 5x - 1}{3x^3 - 5x + 4} = ?$

(a) 4 (b) $\frac{4}{3}$ (c) $\frac{1}{3}$ (d) ∞

32. $\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = ?$

(a) e^{-a} (b) e^a (c) 1 (d) None of these

33. $\lim_{n\to\infty} \left(\frac{1 + 2 + 3 + \cdots\cdots + n}{n^2}\right) = ?$

(a) 0 (b) $\frac{1}{2}$ (c) ∞ (d) None of these

34. $\lim_{n\to\infty} \left(\frac{1^2 + 2^2 + 3^2 + \cdots\cdots + n^2}{n^3}\right) = ?$

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

35. $\lim_{n\to\infty} \left(\frac{1^3 + 2^3 + 3^3 + \cdots\cdots + n^3}{(3n^4 + 5n^3 + 6)}\right) = ?$

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) $\frac{1}{12}$

36. $\lim_{n\to\infty} \frac{n(1^3 + 2^3 + 3^3 + \cdots\cdots + n^3)^2}{(1^2 + 2^2 + 3^2 + \cdots\cdots + n^2)^3} = ?$

(a) $\frac{16}{27}$ (b) $\frac{27}{16}$ (c) ∞ (d) 0

37. $\lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\cdots \frac{1}{3^n} + \cdots\cdots\right) = ?$

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{2}$ (d) None of these

38. $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = ?$

(a) 1 (b) $\frac{1}{2}$ (c) 0 (d) None of these

39. $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)=?$

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2}$ (d) None of these

40. $\lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right)=?$

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) None of these

41. $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+1}=?$

(a) e (b) e^4 (c) e^5 (d) e^x

42. $\lim_{x\to 0}\frac{2}{x}\log(1+x)=?$

(a) ∞ (b) 2 (c) 1 (d) 0

43. $\lim_{x\to 0}\frac{3^x-1}{\sqrt{1+x}-1}=?$

(a) 2 log 3 (b) $\frac{1}{2}\log 2$ (c) $\frac{1}{2}\log 3$ (d) None of these

44. $\lim_{x\to 0}\frac{(x-3)}{|x-3|}=?$

(a) 0 (b) 1 (c) – 1 (d) Does not exist

45. $\lim_{x\to 0}\frac{3x-|x|}{7x-5|x|}=?$

(a) $\frac{3}{2}$ (b) 2 (c) $\frac{1}{6}$ (d) Does not exist

46. $\lim_{x\to 0}\frac{|x|}{x}=?$

(a) 0 (b) 1 (c) – 1 (d) None of these

47. Evaluate: $\lim_{x\to 0} \frac{9^x - 3^x}{4^x - 2^x} = ?$

(a) $\log_2 3$ (b) log 2 (c) log 3 (d) None of these

48. Evaluate: $\lim_{x\to 0} \frac{\log(1 + px)}{e^{3x} - 1} = ?$

(a) $^{p}/_{3}$ (b) p (c) $-{}^{p}/_{3}$ (d) – p

49. Evaluate: $\lim_{x\to 0} \frac{5^x + 3^x - 2}{x} = ?$

(a) log 15 (b) log 5 (c) log 3 (d) None of these

50. Discuss the continuity of the function at x = 1

$$f(x) = \frac{x + 1}{x^{2 + 1}}$$

(a) Continuous (b) Discontinuous (c) Both a and b (d) None of these

51. Discuss the continuity of:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & when\ x \neq 2 \\ -4 & when\ x = 2 \end{cases}$$

(a) Continuous (b) Discontinuous (c) Both a and b (d) None of these

52. Discuss the continuity of:

$$f(x) = \begin{cases} \frac{x^2 - 3x + 2}{x - 2} & when\ x \neq 1 \\ 1 & when\ x = 1 \end{cases}$$

(a) Continuous (b) Discontinuous (c) Both a and b (d) None of these

53. Examine the continuity of the function:

$$f(x) = \begin{cases} 2 + x, \; if \; x < 1 \\ 2 - x, \; if \; x \geq 1 \end{cases}$$

at x = 1

(a) Continuous (b) Discontinuous (c) Both a and b (d) None of these

54. A function is defined as follows:

$$f(x) = \begin{cases} \frac{9x}{x+2}, if \; x < 1 \\ \frac{-x+3}{x}, if \; x \geq 1 \end{cases}$$

Test whether the above function is continuous or not

(a) Continuous (b) Discontinuous (c) Both a and b (d) None of these

55. Let $f(x) = \begin{cases} \frac{x^2}{a} - a, if \; x \leq a \\ a - \frac{a^2}{x}, if \; x > a \end{cases}$

Find the value of R.H.L

(a) 1 (b) 0 (c) a (d) None of these

56. Determine the value of a, so that f(x) is continuous:

$$f(x) = \begin{cases} ax + 5, if \; x \leq 2 \\ x - 1, if \; x > 2 \end{cases}$$

(a) – 2 (b) 2 (c) 1 (d) – 1

57. A function f(x) is defined as follows:

$$f(x) = \begin{cases} \frac{1}{2} - x, if -\frac{1}{2} < x < \frac{1}{2} \\ \frac{3}{2} - x, if \ \frac{1}{2} < x < 1 \\ \frac{1}{2}, if \ x = \frac{1}{2} \end{cases}$$

Find the L.H.L at x = $1/2$

(a) 0 (b) 1 (c) – 1 (d) 2

58. A function f(x) is defined as follows:

$$f(x) = \begin{cases} 3 + 2x, \ if -\frac{3}{2} < x < 0 \\ 3 - 2x, \ if \ 0 < x < \frac{3}{2} \\ -3 - 2x, \ if \ x > \frac{3}{2} \end{cases}$$

f(x) is continuous at x = 0 and is discontinuous at x = 3/2

(a) True (b) False
(c) No conclusion is drawn (d) None of these

59. A function f(x) is defined as:

$$f(x) = \begin{cases} x + 1, if -1 < x < 1 \\ x, if \ 1 < x < 2 \\ 4 - x, if \ 2 < x < 5 \end{cases}$$

Discuss the continuity of the function as x = 1 and x = 2.

(a) Continuous, Discontinuous (b) Continuous, Continuous
(b) Discontinuous, Discontinuous (d) Discontinuous, Continuous

60. If $f(x) = \begin{cases} 3, & if\ x < 1 \\ ax + b, & if\ 1 \leq x < 3 \\ 9, & if\ 3 < x \end{cases}$

Determine the values of a and b, so that f(x) is continuous.

(a) a = 3, b = 0 (b) a = 3, b = 3 (c) a = 0, b = 3 (d) a = 3, b = – 3

ANSWERS						
1 (d)	2 (c)	3 (b)	4 (a)	5 (b)	6 (c)	7 (a)
8 (b)	9 (c)	10 (a)	11 (c)	12 (a)	13 (c)	14 (b)
15 (a)	16 (a)	17 (a)	18 (c)	19 (b)	20 (b)	21 (a)
22 (a)	23 (b)	24 (a)	25 (a)	26 (b)	27 (a)	28 (a)
29 (d)	30 (a)	31 (d)	32 (b)	33 (b)	34 (a)	35 (d)
36 (b)	37 (c)	38 (c)	39 (c)	40 (b)	41 (c)	42 (b)
43 (a)	44 (d)	45 (d)	46 (c)	47 (a)	48 (a)	49 (a)
50 (a)	51 (b)	52 (b)	53 (b)	54 (b)	55 (b)	56 (a)
57 (a)	58 (a)	59 (d)	60 (a)			