Chapter-16

Theoretical Distribution

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Q- If a coin tossed 3 times, X denotes the number of Heads then X is a random variable

$S = \{HHH, HHT, HTT, HTH, TTT, TTH, THH, THT\}$

X = no. of Heads

$X = 0, 1, 2, 3$

$$\left.\begin{aligned} P(X=0) &= 1/8 \\ P(X=1) &= 3/8 \\ P(X=2) &= 3/8 \\ P(X=3) &= 1/8 \end{aligned}\right\} \text{Calculation manually}$$

Formula

$$P(X=x) = {}^nC_x \, p^x q^{n-x}$$

X ↓ Symbol ↓ It can'not be change

x → value ↓ It can change (0, 1, 2, ... etc.)

n = number of trials

x = value of random variable for which probability to be calculated

p = probability of success

q = probability of failure

$\left[{}^{n}C_x \cdot p^x \cdot q^{n-x}\right]$

Same question by formulae →

$n = 3$

$p = \frac{1}{2}$ (occurance of Head)

$q = \frac{1}{2}$ (Non occurance of Head)

$P(x=0) = {}^3C_0 \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^{3-0} = \frac{1}{8}$

$P(x=1) = {}^3C_1 \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^2 = \frac{3}{8}$

$P(x=2) = {}^3C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^1 = \frac{3}{8}$

$P(x=3) = {}^3C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^0 = \frac{1}{8}$

$\Rightarrow P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$\Rightarrow \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$

$\Rightarrow \frac{\cancel{8}}{\cancel{8}} = \underline{\underline{1}}.$

${}^{n}C_x \cdot p^x \cdot q^{n-x}$ applies everywhere

$\times$ $\times$

Binomial poisson | Normal

↓ ↓

Discrete Continous

* Types of probability function

Random variable type	Type of probability function
① Discrete	probability mass function [PMF)
② Continous	probability Density function [PDF]

* Binomial distribution

(bi- parametric discrete probability distribution)

→ Bernoulli's Trial

① Each trial is mutually exclusive & Exhaustive

② Trials are independent

③ probability of success (p) and failure ($q = 1 - p$) will remain unchanged.

④ No. of Trial is positive integer

→ A Random variable that follows Binomial dist. will be called as Binomial variable. [$X \sim B(n, p)$]

Ex - Three coins are tossed,

X = no. of heads [3)

$p = 1/2$

$$\boxed{X \sim B\left(3, \frac{1}{2}\right)}$$

Q- Three coins are tossed, x is random variable denotes no. of Heads

Sol^n

X	$P(X)$	$X \cdot P(X)$	$X^2 \cdot P$
0	$\frac{1}{8}$	$\frac{0}{8}$	$\frac{0}{8}$
1	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$
2	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{12}{8}$
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{9}{8}$
		$12/8$	$24/8 = 3$

$\mu = \Sigma(X) = \frac{12}{8} = 1.5$

$\Sigma X^2 \cdot P = 3$ $\quad [\Sigma x^2 P - (\Sigma x)^2]$

Variance $V(x) = 3 - (1.5)^2$

$= 0.75$

$\boxed{\mu = n \cdot P}$

$\mu = 3 \cdot \times \frac{1}{2} = \frac{3}{2} = 1.5$

$\boxed{\text{Variance} = n \cdot p \cdot q}$

$V(x) = 3 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} = 0.75$

* $n = 8$

Case	p	q	mean (np)	variance (npq)
A	0.10	0.90	0.8	0.72
B	0.20	0.80	1.6	1.28
C	0.30	0.70	2.4	1.68
D	0.40	0.60	3.2	1.92
E	0.50	0.50	4.0	2.0
F	0.60	0.40	4.8	19.2
G	0.70	0.30	5.6	1.68

Observation :-

① mean > variance

② variance will be maximum when $p = q = 0.5$

max value of variance $= npq$

$= n \times \frac{1}{2} \times \frac{1}{2}$

max. value of Variance $= \boxed{\frac{n}{4}}$

* **Mode** - observation with highest frequency / Random variable with highest probability

Coin is tossed 3 times	Coin is tossed 4 times
$X \sim B(3, \frac{1}{2})$	$X \sim B(4, \frac{1}{2})$

X	P
0	$1/8$
1	$3/8$
2	$3/8$
3	$1/8$

Mode = 1, 2 (manually)

X	P
0	$1/16$
1	$4/16$
2	$6/16$
3	$4/16$
4	$1/16$

Mode = 2 (manually)

Formula of mode = $\boxed{\mu = (n+1)P}$

Integer → Two modes → $\boxed{\mu = (n+1)p}$, $\boxed{\mu = (n+1)P - 1}$

Non-Integer → one mode → largest Integer contain in $\boxed{(n+1)P}$

Same Question by formulae

$p = \frac{1}{2}$, $n = 3$	$p = \frac{1}{2}$, $n = 4$
$\mu = (n+1)p$	$\mu = (n+1)p$
$\mu = (3+1)\frac{1}{2}$	$\mu = (4+1)\frac{1}{2}$
$= 4 \times \frac{1}{2} = 2$	$\mu = 5/2 = 2.5$
$\downarrow$ Integer (means 2 modes)	$\downarrow$ Non-Integer (means single mode)
$(n+1)p \to 2$; $(n+1)p-1 \to 2-1 = 1$	$\downarrow$ $(n+1)p$
	$\mu =$ largest Integer in $(n+1)p$ $\to 2.5 \to 2 + 0.5 \to$ $\boxed{\text{mode} = 2}$

* **Additive property**

X and Y are two independent variable such that $X \sim B(n_1, p)$ and $Y \sim (n_2, p)$ then

$$\boxed{(x+y) \sim B(n_1 + n_2, p)}$$

Ex- (I) $x \sim B(6, \boxed{\frac{1}{3}})$ & $y \sim B(3, \boxed{\frac{1}{3}})$

$X+Y \sim B(9, \frac{1}{3})$

(II) $X \sim B(6, \boxed{\frac{1}{3}})$ & $Y \sim B(6, \boxed{\frac{1}{2}})$

Not possible.

Because P is not equal, no matters n are equal or not.

Q- A Coin is tossed 10 times. probability of getting atleast 4 Heads ?

Soln= $\underbrace{X \Rightarrow 4 \text{ or more} \Rightarrow 4, 5, 6, 7, 8, 9, 10}_{\text{very long way}}$

Shorter way :- $n = 10, \; p = \frac{1}{2}$

$$P(X=x) = {}^{n}C_{x} \cdot p^{x} \cdot q^{n-x}$$

$$= {}^{10}C_{x} \cdot p^{x} \cdot q^{10-x}$$

$$= {}^{10}C_{x} \cdot \left(\frac{1}{2}\right)^{x} \cdot \left(\frac{1}{2}\right)^{10-x}$$

$$= {}^{10}C_{x} \cdot \left(\frac{1}{2}\right)^{\cancel{x}+10-\cancel{x}}$$

$$= \boxed{{}^{10}C_{x} \cdot \left(\frac{1}{2}\right)^{10}}$$

$\Rightarrow P(x \geq 4) = P(X=4) + P(x=5) + P(x=6)$
$+ P(x=7) + P(x=8) + P(x=9) + P(x=10)$

$$\Rightarrow \frac{{}^{10}C_4 + {}^{10}C_5 + {}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}}{2^{10}}$$

$$\Rightarrow \frac{210 + 252 + 210 + 120 + 45 + 10 + 1}{1024} = \frac{848}{1024}$$

$\Rightarrow \underline{\underline{0.828}}$

We Can do the same Question as,

$P(\text{atleast 4 head}) = 1 - P(X \leq 3)$

$$= 1 - \left[\frac{P(x=0) + P(x=1) + P(x=2) + P(x=3)}{2^{10}}\right]$$

$$= 1 - \left[\frac{{}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3}{1024}\right]$$

$$= 1 - \left[\frac{1 + 10 + 45 + 210}{1024}\right]$$

$$= 1 - \left[\frac{176}{1024}\right]$$

$$= \frac{1024 - 176}{1024}$$

$$= \frac{848}{1024} = \underline{\underline{0.828}}$$

* Poisson Distribution

Discrete Random Variable — Binomial, Poisson

Continous Random variable → Normal

Reoson why any distribution other than Poisson is needed?

$n \longrightarrow$ High
$p \longrightarrow$ Low
} use poisson

* Introduced by "Simon Denis Poisson" of France in 1837

* Limiting form of Binomial distribution
$n \longrightarrow \infty$
$p \longrightarrow 0$

* Poisson variable denoted as
$X \sim P(m)$

m ↓ Single parameter ↓

That's why it is also known as "Uni-parametric discrete probability distribution".

* Probability mass function

$$\boxed{f(x) = P(X = x) = \frac{(e^{-m} \cdot m^x)}{x!}}$$

for $x = 0, 1, 2, \ldots, \infty$

$\boxed{e = 2.71828}$

$\boxed{m = np}$

* $\boxed{\text{mean } (\mu) = m}$

$\boxed{\text{variance } \sigma^2 = m}$

$\boxed{\text{standard deviation} = \sqrt{m}}$

Mode = Step 1. – find m

Step 2. – If m is integer then (Two modes)

$$\underbrace{\boxed{\mu_0 = m \ \& \ m-1}}_{\text{Bimodal}}$$

Step 3. – If m is non-Integer (one mode)

$$\underbrace{\boxed{\mu_0 = \text{largest integer contained in } m}}_{\text{uni-modal}}$$

* Additive Property

$X \sim P(m_1)$ $Y \sim P(m_2)$ are two independent variables. Then,

$$\boxed{(X + Y) \sim P(m_1 + m_2)}$$

Ex - If 2% of electric bulb mfg. by Company are defective. what is the probability that a sample of 150 bulbs taken would contain:

(i) exactly one defective bulb

(ii) more than 2 defective bulb

Sol^n → (i) $p(X = 1) = ?$

$m = 150 \times 2\%$ $[m = np]$

$\boxed{m = 3}$

$$P(x) = \frac{e^{-3} \cdot (3)^1}{1!}$$

$P(x) = \boxed{0.14936}$

(ii) $P(\text{more than } 2) = 1 - (P(X \leq 2))$

$$= 1 - \left[\frac{e^{-3} \cdot m^0}{0!} + \frac{e^{-3} \cdot m^1}{1!} + \frac{e^{-3} \cdot m^2}{2!}\right]$$

$$= 1 - \left[e^{-3}\left(\frac{1}{1} + \frac{m}{1} + \frac{m^2}{2}\right)\right]$$

$$= 1 - e^{-3}\left(1 + 3 + \frac{9}{2}\right)$$

$$= 1 - e^{-3}(8.5)$$

$= \boxed{0.57680}$

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* Poisson model (Theory)

I. The probability of finding success in a very small time interval $(t, t+dt)$ is Kt, where $(K>0)$ is a constant.

II. The probability of having more than one success in this time interval is very low.

III. The probability of having success in this time interval is indendent of t as well as earlier success.

* NORMAL or GAUSSIAN DISTRIBUTION

Discrete R.V.
- Binomial
- Poisson

Continuous R.V.
- Normal

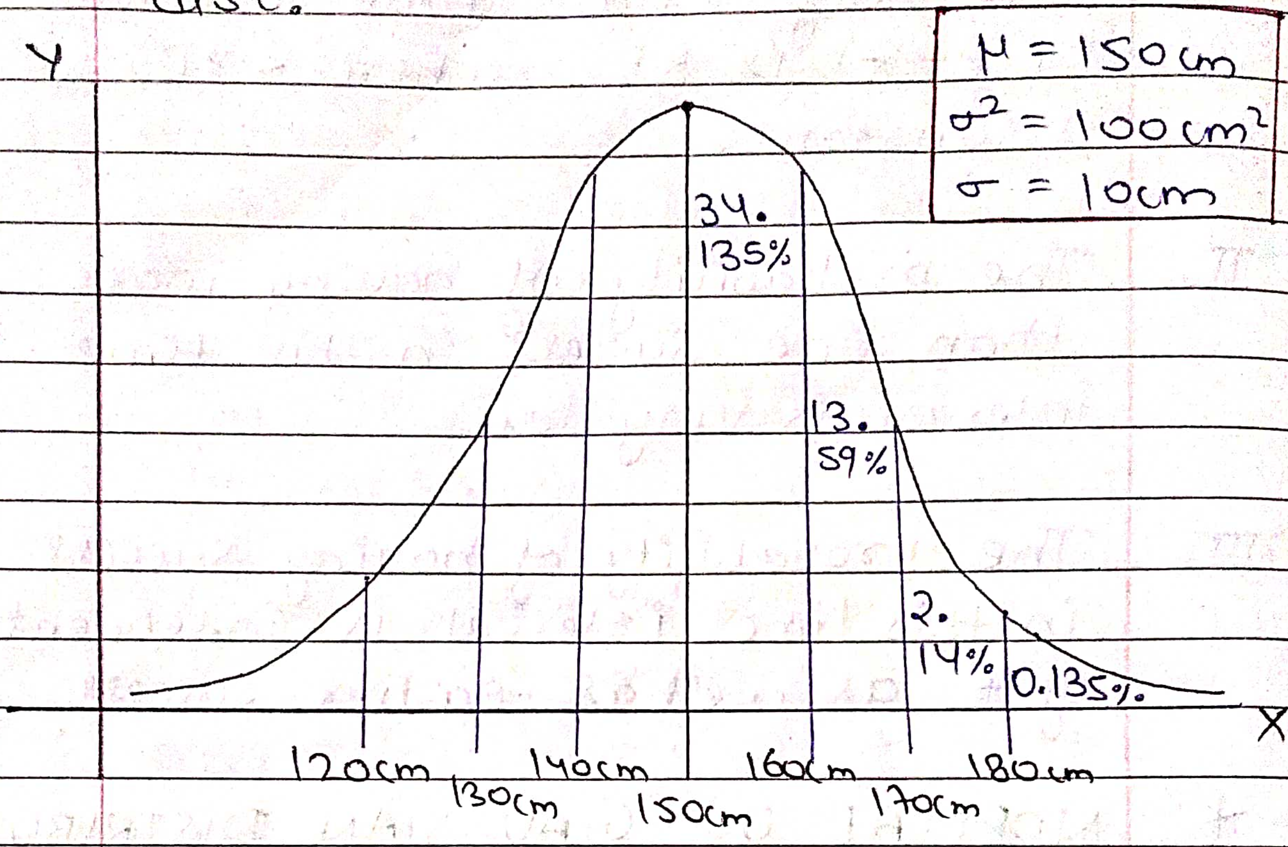
Normal Distribution

Graphical way with Z table → will be used for Calculating probabability

Formula → No use of Calculating probability

Normal dist.:

Most of the continuous random variable follows symmetrical dist.



X = height in cm.

$P(X > 150 cm) = 0.5 = 50\%$

$P(X < 150 cm) = 0.5 = 50\%$

$P(X > 160 cm)$ = less than 0.5 = 15.865%

$P(X < 160 cm)$ = more than 0.5 = 84.135%

$P(X < 130 cm) = 2.275\%$

$P(X > 140 cm) = 84.135\%$

$P(130 < x < 160) = 81.86\%$

↔ Total area under normal curve = 1

↔ Area under Normal curve representing the probability.

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→ probability Density function

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} \times e^{-\left(\frac{x-\mu}{\sigma}\right)^2 \times \frac{1}{2}}}$$

→ Normal variable

$$\boxed{X \sim N(\mu, \sigma^2)}$$

→ $\boxed{\text{Mean} = \text{Median} = \text{Mode} = \mu}$

$\boxed{\text{Standard deviation} = \sigma}$

$\boxed{\text{Mean deviation} \Rightarrow \sigma \times \sqrt{2\pi} = 0.8\sigma}$

Quartiles ⇒ $\boxed{Q_1 = \mu - 0.675\sigma}$

$\boxed{Q_3 = \mu + 0.675\sigma}$

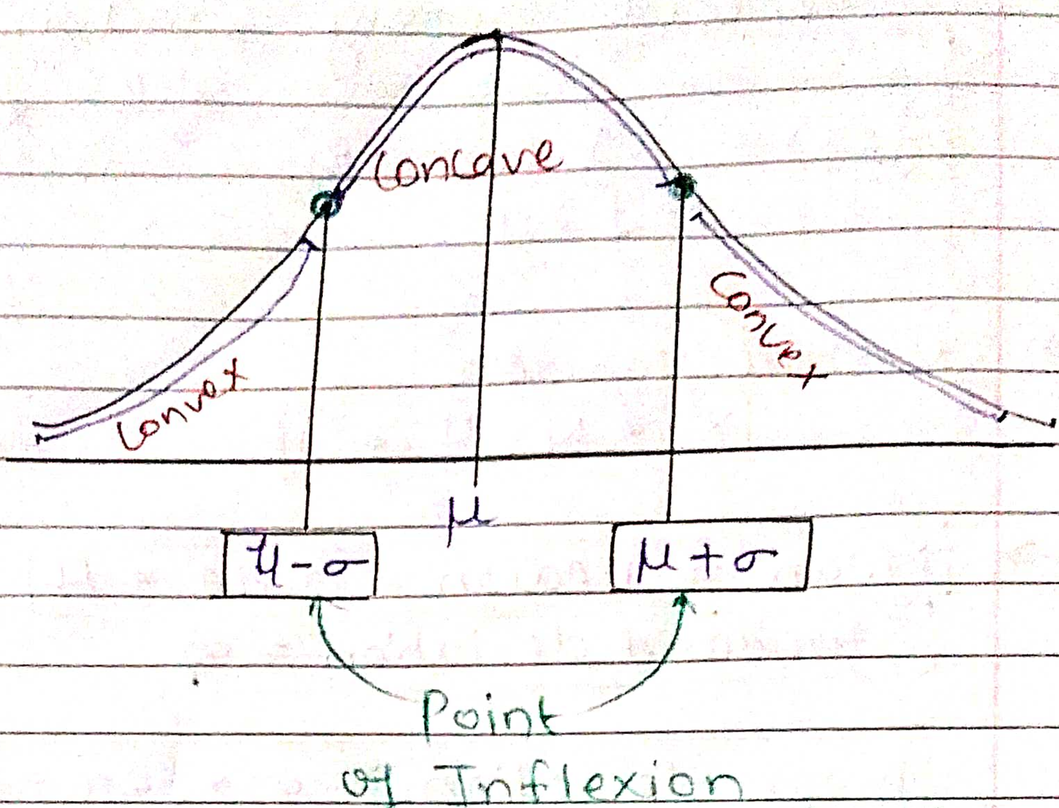
$\boxed{\text{Quartile deviation} \Rightarrow 0.675\sigma}$

$\boxed{QD : MD : SD \Rightarrow 10 : 12 : 15}$

→ Curve formed by Normal distribution Called as probability curve.

→ Tails of the Normal curve never touch the Horizontal axis.

→ It is also known as "bi-paramatric continuous probability distribution.



* STANDARD NORMAL DISTRIBUTION

→ For converting X to Z score =

$$\boxed{Z = \frac{X - \mu}{\sigma}}$$

→ $\boxed{\text{Mean, median, mode} \Rightarrow \mu = 0}$

SD, variance $\sigma = 1$ $\sigma^2 = 1$

point of inflexion = $(-1, 1)$

mean deviation = 0.8

Quartile Deviation = 0.675

→ Z table = Z table gives us the probability of values from $Z = 0$ to any value Z. (Area from center)

→ Cumulative Distribution function

$$\boxed{\phi(k) = P(X \leq k)}$$

$\phi(k)$ refers to the area from $-\infty$ to k in the curve.