SEQUENCE AND SERIES

INTRODUCTION :-

(1.0) SEQUENCE

An arrangement of numbers in a definite order according to some rule is called a sequence. The various numbers occurring in a sequence are called its ems we denote the terms of a sequence by a_1, a_2, a_3 ---- etc. The nth term a_n is also called general term.

A sequence is said to be finite or infinite according as it has finite or infinite terms.

Examples :-

(1) 1, 3, 5, 7 ---- is an infinite sequence whose nth term is given by the formula $a_n = 2n-1$, where n is a natural number.

(2) $1, 1/2, 1/3, 1/4$ ----- is an infinite sequence where n^{th} term is given by the formula $a_n = 1/n$, where n is a natural number.

(3) 2, 4, 6, 8, 10, 12 is a finite sequence in which each term is obtained by adding 2 to the previous term.

(2.0) Arithmetic Progression (A.P.)

1. General Term of an A.P. : If a is the first term and d is the common difference of an A.P. then its n^{th} term t_n is given by $t_n = a+(n-1)d$.

2. Sum of n Terms of an A.P. : The sum S_n of first n terms of an A.P. with first term a and common difference d is given by- $S_n = \frac{n}{2}[2a+(n-1)d]$

or, $S_n = \frac{n}{2}[a+l]$, where l is the last term $a+(n-1)d$.

Tutorial Notes

1. If the sum S_n of first n terms of a sequence is given, then the n^{th} term, a_n of the sequence can be determined by $a_n = S_n - S_{n-1}$.

2. If the sum of first n terms of an A.P. is of the form $An^2 + Bn$, where A and B are constant, then common difference is $2A$.

(3) Properties of A.P.

1. If a constant is added or subtracted from each term of an A.P., then the resulting sequence is also in A.P., with same common difference.

2. If each term of an A.P. is multiplied or divided by a non-zero constant K, then the resulting sequence is also an A.P. with common difference Kd or $\frac{d}{K}$, where d is the common difference of the given A.P.

3. If $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3 \dots$ are two arithmetic Progressions, then the sequence $a_1+b_1, a_2+b_2, a_3+b_3 \dots$ is also an A.P.

4. In a finite A.P the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and ~~test~~ last term.
i.e., $a_K + a_{n-(K-1)} = a_1 + a_n$, $K = 1, 2, 3, \dots n-1$.

5. Three numbers a, b, c are in A.P. if $2b = a+c$.

(4.) Sum of n Arithmetic means between a and b $= \frac{n}{2}(a+b)$

(3.0) GEOMETRIC PROGRESSION (G.P)

General term of G.P. :- If a is the first term and r is the common ratio of G.P. then the terms of G.P. are $a, ar, ar^2, ar^3 \dots$ and the n^{th} term t_n is given by -
$t_n = ar^{(n-1)}$

Sum of n terms of a G.P. :-

The sum of first n terms of a G.P with first term a and common ratio r is given by

$$S_n = \begin{cases} \frac{a(r^n-1)}{r-1} & \text{if } r \neq 1 \\ na & \text{if } r = 1 \end{cases}$$

Sum of an infinite G.P. :- The sum of an infinite G.P with first term a and common ratio r ($|r|<1$) is

$$S = \frac{a}{1-r}$$

Geometric Mean

(a) Single Geometric mean :- A number G is said to be the geometric between two non-zero numbers a and b if a, G, b are in G.P. i.e. if $G^2 = ab$.

For Example :- since G 2, 4, 8 are in G.P., therefore 4 is G.M at between 2 and 8.

(b) Non-Geometric Mean :- The number, $G_1, G_2 \cdots G_n$ are said to be n-geometric means between two non-zero numbers a and b if $a, G_1, G_2, \cdots G_n, b$ are in G.P.

Tutorial Note :-

- The product of the n Geometric means is $(ab)^{n/2} = (\sqrt{ab})^n$

4.0 HARMONIC PROGRESSION (H.P)

A sequence of non-zero numbers $a_1, a_2, a_3 \cdots$ is said to be a harmonic progression if the sequence $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \cdots$ in an A.P.

For Example, The sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7} \cdots$ is a H.P because the sequence $1, 3, 5, 7 \cdots$ is an A.P.

A general HP is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$ ....

(5.0) Some Special Sequences

1. The Sum of the first n Natural numbers

$$\Sigma n = 1+2+\cdots+n = \frac{n(n+1)}{2}$$

2. The Sum of the Squares of first n Natural Numbers

$$\Sigma n^2 = 1^2+2^2+\cdots\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. The Sum of the Cubes of first n Natural Numbers

$$\Sigma n^2 = 1^3+2^3+\cdots\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

4. If the nth term of a Sequence is $T_n = An^3+Bn^2+Cn+D$ Then the Sum of n terms is given by

$$\Sigma n = \Sigma T_n = A\Sigma n^3 + B\Sigma n^2 + C\Sigma n + nD$$

ARITHMETIC & GEOMETRIC

A.P

Series having Equal difference is Known A.P

3, 7, 11, 15, 19, 23

$d = T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = \cdots\cdots = T_n - T_{n-1}$

$a = 3 \quad d = 4$

A.P

$T_1 = a$

$T_2 = a+d$

$T_3 = a+2d$

$T_4 = a+3d$

$T_5 = a+4d$

$T_n = a+(n-1)d$

G.P

Series having Equal ratio is known as G.P

1, 4, 16, 64

$R = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_5} = \frac{T_n}{T_{n-1}}$

$a = 1 \quad r = 4$

G.P

$T_1 = a$

$T_2 = a.r$

$T_3 = a.r^2$

$T_4 = a.r^3$

$T_5 = a.r^4$

$T_n = a.r^{n-1}$

Rules of A.P. G.P.

(1) The n^{th} term = last term

$T_n = l$

$T_n = a+(n-1)d$

$T_{10} = a+9d$

$T_5 = a+4d$

$T_{100} = a+99d$

(2) Sum of first 'n' term (S_n)

$S_n = \frac{n}{2}[2a+(n-1)d]$

→ When last term (l) is given in question

$S_n = \frac{n}{2}(a+l)$

(3) Find out value of 'n'

When $a = d$

$n = \frac{l}{a}$

When $a \neq d$

$n = \frac{l-a}{d}+1$

Proof:

$l = a+(n-1)d$

$\frac{l-a}{d}+1 = n$

$n = \frac{l-a}{d}+1$

$n = \frac{l-a}{d}+1$

$n = \frac{l-a}{a}+1$

$n = l/a$

(4) ⇒ Sum of 1^{st} 'n' odd no. $= n^2$

$1+3+5+7+9$----- 'n' terms

$a = 1$ $d = 2$

Proof:

$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$= \frac{n}{2}\left[2+(n-1)2\right]$

$= \frac{n}{2}\left[2+2n-2\right]$

$\frac{n}{2}(2n)$

$= n^2$

(5)

$\Rightarrow$ Sum of 1st 'n' even no = $n(n+1)$

$2+4+6+8+ \cdots 2n$

$a = 2 \quad l = 2n$

$S_n = n/2\,(a+l)$

$= n/2 \times (2+2n)$

$= n/2 \times 2\,(1+n)$

$= n(n+1)$

(6) Sum of 1st 'n' natural no.

$\Sigma n = \dfrac{n(n+1)}{2}$

$1+2+3 \cdots + n$

Proof :

$S_n = \dfrac{n}{2}(a+l)$

$a = 1 \quad l = n$

$= \dfrac{n}{2}(1+n)$

Σ

$= \dfrac{n(n+1)}{2}$

(7) $\Rightarrow$ Sum of squares of 1st 'n' natural No.

ex : $1^2 + 2^2 + 3^2 \cdots + n^2$

$\Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(8) $\Rightarrow$ Sum of cubes of 1st 'n' natural No.

$\Sigma n^3 = \dfrac{n^2(n+1)^2}{4} = \left(\dfrac{n(n+1)}{2}\right)^2 = (\Sigma n)^2$.

(9) $S_m = n \quad S_n = m$

$S_{m+n} = -(m+n)$

Eg :-

① $S_{40} = 20 \quad S_{20} = 40$

$S_{60} = 60$

② $S_{100} = 40 \quad S_{40} = 100 \quad S_{140} = -140$

$S_n = n/2\,(2a + 99d)$

$40 = 50\,(2a + 99d)$

4

Scanned with CamScanner

$2a+99d = 0.8 \rightarrow (i)$

$100 = 20(2a+39d)$

$2a+39d = 5 \ (ii)$

from Equation (i) & (ii)

$$S_{140} = \frac{140}{2}\left[2(3.865)+139(-0.07)\right]$$

$60d = 4.2$ $\quad = 70[7.73 - 9.73]$

$d = -0.07$

$= 70(-2)$

$= -140$

$2a+39(-0.07) = 5$

$2a = 5+2.73$

$a = 3.865.$

(10) $\Rightarrow T_m = n \qquad T_n = m$

$T_r = m+n-r$

Eg. $T_x = m+n-x$

$T_p = m+n-P$

Eg. $T_{30} = 20 \quad T_{20} = 30$

$T_{15} = 30+20-15 = 35$

or

Eg. $T_{60} = 20 \quad T_{20} = 60$

$T_{40} = 20+60-40 = 40$

Proof:

$a+59d = 20 \cdots\cdots (i)$

$a+19d = 60 \cdots\cdots (ii)$

$40d = -40$

$d = -1$

$a+59(-1) = 20$

$a-59 = 20$

$a = 20 + 59$

$a = 79$

$T_{40} = a + 39d$

$= 79 + 39(-1)$

$= 40$

$\Rightarrow T_m = n \qquad T_n = m$

$T_{m+n} = m+n-(m+n)$

$= 0$

$\Rightarrow T_{m-n} = m+n-(m-n)$

$= m+n-m+n$

$= 2n$

(11) $T_m = \frac{1}{n} \qquad T_n = \frac{1}{m}$

$S_{mn} = \frac{1}{2}(mn+1)$

$T_{mn} = 1$

(12) $\Rightarrow$ when S_n is given & we find out T_n

$T_n = S_n - S_{n-1}$

$= [T_1 + T_2 + T_3 \cdots\cdots T_{n-1} + T_n] - [T_1 + T_2 + T_3 \cdots T_{(n-1)}]$

$= T_n$

(13) When T_n is given & we find out S_n

$S_n = \Sigma T_n$

$T_1 + T_2 + T_3 \cdots\cdots T_n = S_n = \Sigma T_n$

(14) Arithmetic mean

$A.M = \frac{a+b}{2}$

Eg:- 1, 3, 5, 7 are in A.P

$3 = \frac{1+5}{2} \qquad 5 = \frac{3+7}{2}$

$3 = 6/2 \qquad 5 = 10/2$

$3 = 3 \qquad 5 = 5$

(15) Geometric Mean

$G.M = \sqrt{ab}$

or

$(G.M)^2 = ab$

Eg:- 1, 2, 4, 8, 16 are in G.P

$8 = \sqrt{4 \times 16}$

$(2)^2 = 1 \times 4 \qquad 8 = \sqrt{64}$

$4 = 4 \qquad 8 = 8$

(16.) When ratio of S_n is given & we find Ratio of T_n

$n = 2 \times \text{term} - 1$

(17.) Sum of 3 no. in A.P. = 3a [a-d, a, a+d]

(18.) Sum of 4 no. in A.P. = 4a [a-3d, a-d, a+d, a+3d]

(19.) Sum of 5 no. in A.P. = 5a [a-2d, a-d, a, a+d, a+2d]

(20.) Sum of squares of 3 no. in A.P. = $(3a^2+2d^2)$

(21.) Sum of squares of 4 no. in A.P. = $(4a^2+20d^2)$

(22.) Sum of squares of 5 no. in A.P. = $(5a^2+10d^2)$

Note :- When we have to find out numbers, always use trial & error method.

otherwise always use actual method.

Geometric Progressions

(1) Last term = nth term. $T_n = l = a \cdot r^{n-1}$

$T_{10} = a \cdot r^9$

$T_3 = a \cdot r^2$

$T_8 = a \cdot r^7$

(2) Sum of 'n' term

When	When	Where
$r = 1$	$r > 1$	$r < 1$
$S_n = n \cdot a$	$S_n = \frac{a(r^n - 1)}{r-1}$	$S_n = \frac{a(1-r^n)}{1-r}$

E.g. 5+5+5 ...... 100 terms

$a = 5 \quad r = 1$

$S_n = n \cdot a$

$= 100 \times 5$

$= 500$

(3) Sum of Infinite terms in G.P.

$T_1 + T_2 + T_3 \ldots\ldots \infty$

$a + ar + ar^2 \ldots\ldots \infty$

$S_\infty = \frac{a}{1-r}$ when $-1 < r < 1$

$\Rightarrow$ $r > 1 \Rightarrow r^\infty = \infty$

$r = 2$ $\quad 2^1 = 2$ $\quad 2^{10} = 1024$

$2^3 = 4$ $\quad 2^\infty = \infty$

$2^3 = 8$

$\Rightarrow$ when $r < 1 \Rightarrow r^\infty = 0$

$r = 1/2$ $\quad (1/2)^1 = 0.50$

$(1/2)^2 = 0.25$ $\quad = (1/2)^\infty = 0$

(4) Sum of squares of infinite terms

$T_1^2 + T_2^2 + T_3^3 \ldots\ldots T_\infty^2$

$a^2 + a^2r^2 + a^2r^4 \ldots\ldots \infty$

$S_\infty^2 = \frac{a^2}{1-r^2}$ when $-1 < r < 1$

(5) Sum of cubes of infinite terms of G

$T_1^3 + T_2^3 + T_3^3 \ldots\ldots T_\infty^3$

$S_\infty^3 = \frac{a^3}{1-r^3}$ when $-1 < r < 1$

(6) $T_a = X$ $\quad T_{a+d} = Y$ $\quad T_{a+2d} = Z$

a, a+d, a+2d are in A.P

then

X, Y, Z are in G.P

$Y^2 = XZ$

Eg:- 1, 2, 4, 8, 16, 32, 64, 128

$T_2 = 2$ $T_5 = 16$ $T_8 = 128$

2, 5, 8 in A.P

2, 16, 128, in G.P

Eg. $T_5 = X$ $T_{10} = Y$ $T_5 = Z$

5, 10, 15 in A.P

x, y, z in G.P

$y^2 = XZ$

(7) $T_{p+q} = m$ $T_{p-q} = n$

$T_p = \sqrt{mn} = (mn)^{1/2} = (mn)^{0.5}$

<u>Proof</u>:- T_1 T_2 T_3 T_4 T_5 T_6 T_7 T_8......

1, 3, 9, 27, 81, 243, 729

$T_{4+3} = T_7 = 729$ $T_{4-3} = T_1 = 1$

$T_{p+q} = m$ $T_{p-q} = n$

$T_4 = \sqrt{mn} = \sqrt{729 \times 1} = 27$

(8) S = sum of first 'n' terms of G.P

P = Product of first 'n' terms of G.P

R = sum of Resiprocals of first 'n' terms of G.P

$P^2 \times R^n = S^n$

Proof: $S = T_1 + T_3 + T_3 \cdots + T_n$

$P = T_1 \times T_2 \times T_3 \cdots \times T_n$

$R = \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} \cdots \frac{1}{T_n}$

$P^2 \times R^n = S^n$

(9) 3 No. in G.P $\frac{a}{r}, a, ar$

(10) 4 No. in G.P $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

(11) 5 No. in G.P. $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

Note : product of No. must be given in question

⇒ Power series

- Series not being A.P & G.P is Known as power series
- Always use trial & Error Method in case of power series
- Always taken $n=2$

⇒ $AM \geq G.m$ (2 positive no.)

$a=b$ | $a \neq b$
$A.m = G.m$ | $A.m > G.m$