PROBABILITY

- Probablity
The terms "probably", 'chance', 'odds in favour', 'odds against' are too familiar nowadays and they have their origin in a branch of mathematics, known as probability.

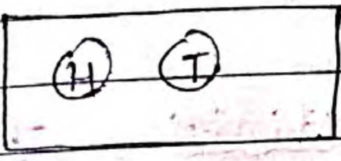
In order to develop a sound knowledge about probability, it is necessary to get ourselves familiar with a few terms.

- Experiment
- Random experiment eg. coin

  Head Tails

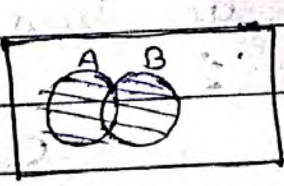
- Events :-
- Simple :- Head | Tail
- Composite :- HT/TH if two coins tossed together.

* **Mutually Exclusive Events or Incompatible Events :-**



like disjoint sets.

* **Exhaustive Events :-**



$n(U) = n(A \cup B)$

* **Equally Likely Events :-**

* Classical definition of probability or a priori definition

* For finite elementary events :-

$$\boxed{P(A) = \frac{n_A}{n} = \frac{\text{No. of equally likely events favourable to } A}{\text{Total no. of equally likely events}}}$$

eg. $P(1 \text{ on dice}) = \frac{1}{6}$

* For finite composite events :-

$$\boxed{P(A) = \frac{m_A}{m} = \frac{\text{No. of mutually exclusive, exhaustive and equally likely events favourable to } A}{\text{Total no. of mutually exclusive and equally likely events.}}}$$

eg. $P(\text{even no. when dice thrown}) = \frac{3}{6} = \frac{1}{2}$

– In connection with classical definition of probability we may note the following points :-

a) The probability of an event lies b/w 0 and 1, both inclusive ($0 \leq P(x) \leq 1$)

eg. $P(7 \text{ when dice thrown}) = \frac{0}{6} = 0$

$P(\text{getting no. less than 7 when dice thrown}) = \frac{6}{6} = 1$

b) Non-occurrence of event A is denoted by A' and it is known as complementary event of A. The event A along with its complimentary A' forms a set of mutually exclusive and exhaustive events.

$$P(A) + P(A') = 1$$

c) The ratio of no. of favourable events to the no. of unfavourable events is known as odds in favour of the event. A and its inverse ratio is known as odds against the event A.

$n_A \rightarrow$ no. of events favour in A.

$n_B \rightarrow$ no. of events against A.

- Odds in favour $= n_A : n_B$

$$P(A) = \frac{n_A}{n_A + n_B}$$

- Odds in against $= n_B : n_A$

$$P(B) = \frac{n_B}{n_A + n_B}$$

* **Demerits or limitations**

1) Applicable when the total no of events is finite.

2) Used when the events are equally likely or equi-probable

3) It has limited field of appn like coin tossing, dice throwing, cards etc.

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* Relative Frequency definition of Probability

Let us consider a random experiment repeated a very good number of times, say n, under an identical set of conditions.

We next assume that an event A occurs f_A times. Then the limiting value of the ratio of f_A to n as n tends to infinity is defined as the probability of A.

$$\boxed{P(A) = \frac{f_a}{n}}$$ f_a → no. of favourable event to A. n → Total no. of event

* Operations on events - sets theoretic approach to probability

Total no. of event $= n(S)$, S :- Sample space

Then, $\boxed{P(A) = \frac{n(A)}{n(S)}}$

- Two events A and B are mutually exclusive if $P(A \cap B) = 0$

And $P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{n(A)+n(B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)}$

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

- Similarly three events A, B and C are mutually exclusive if

$$\boxed{P(A \cup B \cup C) = P(A) + P(B) + P(C)}$$

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$P(A \cap B) = 0$
$P(B \cap C) = 0$
$P(A \cap C) = 0$
$P(A \cap B \cap C) = 0.$

- Two events A and B are exhaustive if

$$n(A \cup B) = n(S)$$

$$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{n(S)}{n(S)} = 1$$

- Similarly three events A, B and C are exhaustive if $\boxed{P(A \cup B \cup C) = 1}$

- Three events A, B and C are equally likely if

$$\boxed{P(A) = P(B) = P(C)}$$

* Axiomatic or modern definition of probability

P(A) is defined as the probability of A if P satisfies the following axioms:

(i) P(A) lies between 0 and 1 for every $A \subseteq S$

$$0 \leq P(A) \leq 1$$

$P(A) = 0$ when no event occurs in A
$P(A) = 1$ when all event occurs in A.

(ii) $P(S) = \frac{n(S)}{n(S)} = 1$

$P(A) = 0 \Rightarrow$ impossible event
$P(A) = 1 \Rightarrow$ sure event

(iii) For any sequence of mutually exclusive events $A_1, A_2, \ldots$

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

* Addition theorems or theorems on total probability *

* Theorem 1

For any two mutually exclusive events A and B, the probability that either A or B occurs is given by the sum of individual probabilities of A and B.

i.e $\boxed{P(A \cup B) \text{ or } P(A+B) = P(A) + P(B)}$

* Theorem 2

For any $K (\geq 2)$ mutually exclusive events $A_1, A_2, A_3, \ldots A_K$ the probability that at least one of them occurs is given by the sum of the individual probabilities of the K events.

i.e. $P(A_1 \cup A_2 \cup \ldots \cup A_K) =$
i.e. $P(A_1) + P(A_2) + P(A_3) + P(A_4) + \ldots$

* Theorem 3

For any two events A and B, the probability

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that either A or B occurs is given by

i.e. $$P(A \cup B) = \frac{n(A \cup B)}{n(S)}$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

* Theorem 4

For any three events A, B and C, the probability that at least one of the events occurs

$$P(A \cup B \cup C) = \frac{n(A \cup B \cup C)}{n(S)}$$

$$= \frac{n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)}{n(S)}$$

$$\boxed{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)}$$

* Compound Probability or Joint Probability

The probability of occurence of two events A and B simultaneously is known as the Compound Probability or Joint Probability of the events A and B

P(A and B) :- Compound Probability / Joint probability.

If the occurrence of one event, say B, is influenced by the occurrence of another event A, then the two events A and B are known as dependent events.

$P(A) \rightarrow$ independent

$P(B/A) \rightarrow$ B is dependent on A

$$P(A \cap B) = P(A) \times P(B/A)$$

$$\boxed{P(B/A) = \frac{P(A \cap B)}{P(A)}}$$

* Compound Probability of dependent events

If we take two successive draws as two event A and B, then finding the probability of second draw is dependent on first draw.

$$\boxed{P(B/A) = \frac{P(A \cap B)}{P(A)}} \qquad P(A \cap B) = P(A) \cdot P(B/A)$$

Similarly;

$$P(A \cap B) = P(A/B) P(B)$$

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$

* Compound Probability of Independent Events:

$P(A \cap B) = P(A) \cdot P(B)$

$P(B/A) = P(B)$
$P(A/B) = P(A)$

* Note:

Whenever not given in que that events are dependent, we assume it is independent.

* $\boxed{P(A \cup B \cup C) = 1}$

* $\boxed{P(A)\, P(B)\, P(C) = P(A \cap B \cap C) = 0}$

* Compound Probability or Joint Probability

Also If two events A and B are independent, then the following pairs of events are also independent:

- $P(A \text{ and } B') = P(A \cap B') = P(A)\, P(B')$

$\therefore \boxed{P(A \text{ and } B') = P(A) \cdot (1 - P(B))}$

- $P(A' \text{ and } B) = P(A' \cap B) = P(A')\, P(B)$

$\therefore \boxed{P(A' \text{ and } B) = [1 - P(A)]\, P(B)}$

- $P(A' \text{ and } B') = P(A' \cap B') = P(A')\, P(B')$

$\therefore \boxed{P(A' \text{ and } B') = [1 - P(A)]\, [1 - P(B)]}$

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* Compound Probability of Three Events

- If there are three A, B and C then the probability of simultaneous events which is dependent on each other will be

$$P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B)$$

$\Downarrow$

[A is independent / B is dependent on A / C is dependent on A and B]

- $\boxed{P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)}$

- $\boxed{P(A \cap B) = P(A) \cdot P(B)}$

- $\boxed{P(A \cup B \cup C) = P(\text{Total}) - P(A' \cap B' \cap C')}$

* Random Variable - Probability Distribution

- A random variable or stochastic variable is a function defined on a sample space associated with a random experiment assuming any value from R (Real no).

- It is done by assigning a real number to each and every sample point of the random experiment.

- A random variable is denoted by a capital letter - X

Example:-

If a coin is tossed three times and if x denotes the number of heads, then x is a random variable

All possible scenarios of choosing different no. of heads:-

1	2	3
H	H	H
H	T	H
H	H	T
H	T	T
T	H	H
T	T	H
T	H	T
T	T	T

Random Variable $\rightarrow X$

$p(X) =$	0	1	2	3
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$p(X) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1.$$

* Discrete Random Variable :-
  eg. no. of accidents, salary credited.

* Continuous Random Variable :-
  eg. height, weight.

• If a random variable X assumes n finite values $X_1, X_2, X_3, ----, X_n$ with corresponding probabilities $P_1, P_2, P_3, -----P_n$ such that

i) $P_i \geq 0$ (for every i)

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ii) $\Sigma P_i = 1$

$\left(\Sigma P(x) = P(X_1) + P(X_2) + P(X_3) + \cdots + P(X_n) = 1\right)$

* Probability Density Function (PDF)

- When x is a continuous random variable defined over an interval $[\alpha, \beta]$, where $\beta > \alpha$, then x can assume an infinite number of values from its interval.

$$X \in [\alpha, \beta]$$

Discrete	Continuous
$X_1, X_2, X_3, X_4 \cdots X_n$	$X \in [\alpha, \beta]$
$P(X_1), P(X_2), \cdots P(X_n)$	$f(X) \Rightarrow$ probability density function $X \in [\alpha, \beta]$
$P(x_i) \geq 0$	$f(x) \geq 0$
$\Sigma P(X) = 1$	$\int_{\alpha}^{\beta} f(x)\,dx = 1$

* Expected value of a Random Variable

If a random variable x assumes n values $x_1, x_2, x_3, \cdots x_n$ with corresponding probabilities $p_1, p_2, p_3 \cdots p_n$ where p_i satisfies

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i) $p_i \geq 0$ for every x_i
ii) $\Sigma p_i = 1$

then the expected value of x is given by
$\mu \Rightarrow E(x) = \Sigma p_i x_i$

Expected value of x^2 is given by E
$E(x^2) = \Sigma p_i x_i^2$

In particular expected value of a monotonic function $g(x)$ is given by...

$$\boxed{E[g(x)] = \Sigma p_i [g(x_i)]}$$

$V(x) = \sigma^2 = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2$
$$\boxed{\sigma^2 = E(x^2) - [E(x)]^2}$$

$S.D = \sigma = \sqrt{\Sigma p_i x_i^2 - (\Sigma p_i x_i)^2}$
$$\boxed{\sigma = \sqrt{E(x^2) - [E(x)]^2}}$$

* Properties.
- If $y = a + bx$, for two random variables x and y and for a pair of constants a and b, then the mean i.e. expected value of y:-
$\mu_y = a + b\mu_x$

and the standard deviation of y is
$\sigma_y = |b| \sigma_x$

and for variance

$\sigma_y^2 = b^2 \sigma_x^2$

- Now, when x is discrete random variable & $f(x)$ is the probability mass function, then the expected value (Mean) :-

P_i or $P(x)$ of $f(x)$

$$\boxed{\mu = E(x) = \Sigma p_i x_i = \Sigma f(x) \cdot x}$$

where $f(x)$ = probability mass function

$$\boxed{E(x^2) = \Sigma p_i x_i^2 = \Sigma f(x) \cdot x^2}$$

It's variance is given by :

$$\boxed{\sigma^2 = E(x^2) - [E(x)]^2 \quad \text{OR} \quad E[x_i - E(x_i)]^2}$$

- And, if x is continuous random variable defined in $[-\infty, \infty]$, then the expected value is

$$\boxed{\mu = E(x) = \int_{-\infty}^{\infty} [f(x) x] \, dx}$$

$$\boxed{E(x^2) = \int_{-\infty}^{\infty} f(x) \cdot x^2 \, dx}$$

and $\boxed{\sigma^2 = E(x^2) - [E(x)]^2}$

* PROPERTIES OF EXPECTED VALUE *

1. Expectation of all value equal to constant k is

$$\boxed{\begin{aligned} E(x) &= \mu = k \\ \sigma &= 0 \\ \sigma^2 &= 0^2 = 0 \end{aligned}}$$

2. Expectation of sum of two random variables is the sum of their expectations

i.e. $$\boxed{\begin{aligned} E(x+y) &= \Sigma p(x+y) \\ &= \Sigma px + \Sigma py \\ &= E(x) + E(y) \end{aligned}}$$

3. Expectation of the product of a constant & a random variable is the product of the constant and the expectation of the ~~var~~ random variable.

i.e. $$\boxed{\begin{aligned} E(kx) &= \Sigma p_i(kx) \\ &= k\Sigma px \\ &= k[E(x)] \end{aligned}}$$

4. Expectation of the product of the two random variables is the product of the expectation of the two random variables, provided the two variables are independent.

i.e. $$\boxed{E(xy) = E(x) \cdot E(y)}$$

$$\boxed{E(x-y) = E(x) - E(y)}$$

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