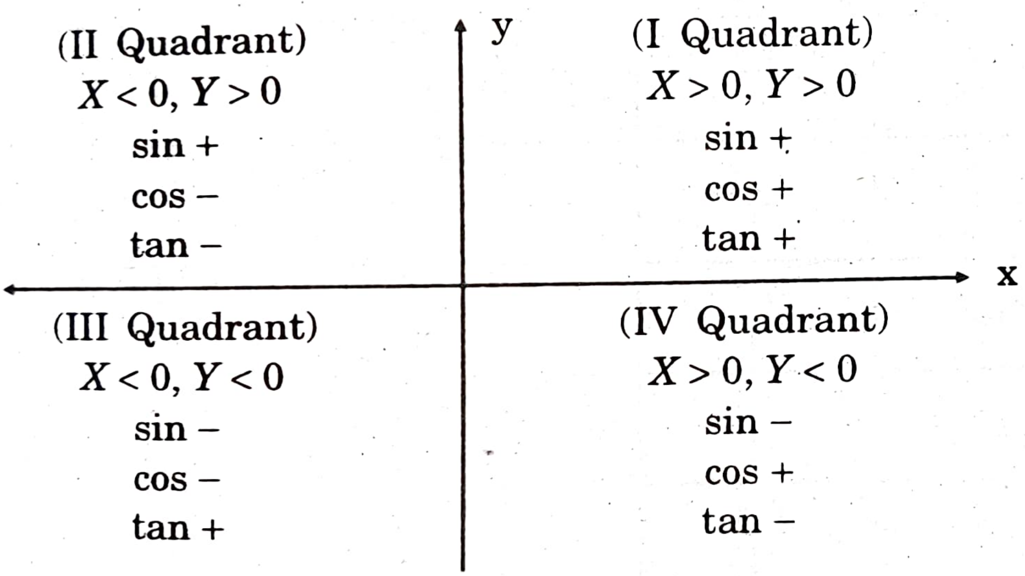
CHAPTER 5

TRIGONOMETRY

5.1. QUADRANTS AND TRIGONOMETRIC RATIOS:



The following are the values of the trigonometric ratios for some specific angles:

θ^0	θ radian	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	∞

θ^0	θ radian	$\sin\theta$	$\cos\theta$	$\tan\theta$
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
135°	$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
180°	π	0	-1	0
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
225°	$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
270°	$\frac{3\pi}{2}$	-1	0	$-\infty$
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
315°	$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$

5.2. COMPLEMENTARY ANGLE:

If θ is an acute angle, then $(90^\circ - \theta)$ is called its complementary angle.

If the sum of two angles is 90° then the two angles are called complementary to each other.

1. $\sin(90^\circ - \theta) = \cos\theta$
2. $\cos(90^\circ - \theta) = \sin\theta$

3. $\tan(90^\circ - \theta) = \cot\theta$
4. $\cot(90^\circ - \theta) = \tan\theta$
5. $\operatorname{cosec}(90^\circ - \theta) = \sec\theta$
6. $\sec(90^\circ - \theta) = \operatorname{cosec}\theta$

Trigonometric Ratios for Angles > 90°:

II Quadrant:

1. $\sin(90^\circ + \theta) = \cos\theta$
2. $\cos(90^\circ + \theta) = -\sin\theta$
3. $\tan(90^\circ + \theta) = -\cot\theta$
4. $\operatorname{cosec}(90^\circ + \theta) = \sec\theta$
5. $\sec(90^\circ + \theta) = -\operatorname{cosec}\theta$
6. $\cot(90^\circ + \theta) = -\tan\theta$
7. $\sin(180^\circ - \theta) = \sin\theta$
8. $\cos(180^\circ - \theta) = -\cos\theta$
9. $\tan(180^\circ - \theta) = -\tan\theta$
10. $\operatorname{cosec}(180^\circ - \theta) = \operatorname{cosec}\theta$
11. $\sec(180^\circ - \theta) = -\sec\theta$
12. $\cot(180^\circ - \theta) = -\cot\theta$

III Quadrant:

1. $\sin(180^\circ + \theta) = -\sin\theta$
2. $\cos(180^\circ + \theta) = -\cos\theta$
3. $\tan(180^\circ + \theta) = \tan\theta$
4. $\operatorname{cosec}(180^\circ + \theta) = -\operatorname{cosec}\theta$
5. $\sec(180^\circ + \theta) = -\sec\theta$
6. $\cot(180^\circ + \theta) = \tan\theta$
7. $\sin(270^\circ - \theta) = -\cos\theta$
8. $\cos(270^\circ - \theta) = -\sin\theta$
9. $\tan(270^\circ - \theta) = \cot\theta$

10. $\operatorname{cosec}(270^\circ - \theta) = -\sec\theta$
11. $\sec(270^\circ - \theta) = -\operatorname{cosec}\theta$
12. $\cot(270^\circ - \theta) = \tan\theta$

IV Quadrant:

1. $\sin(270^\circ + \theta) = -\cos\theta$
2. $\cos(270^\circ + \theta) = \sin\theta$
3. $\tan(270^\circ + \theta) = -\cot\theta$
4. $\operatorname{cosec}(270^\circ + \theta) = -\sec\theta$
5. $\sec(270^\circ + \theta) = \operatorname{cosec}\theta$
6. $\cot(270^\circ + \theta) = -\tan\theta$
7. $\sin(360^\circ - \theta) = -\sin\theta$
8. $\cos(360^\circ - \theta) = \cos\theta$
9. $\tan(360^\circ - \theta) = -\tan\theta$
10. $\operatorname{cosec}(360^\circ - \theta) = -\operatorname{cosec}\theta$
11. $\sec(360^\circ - \theta) = \sec\theta$
12. $\cot(360^\circ - \theta) = -\cot\theta$

Trigonometric ratios for Negative Angles:

Positive angles are measured in the anticlockwise direction. Negative angles are measured in the clockwise direction.

1. $\sin(-\theta) = -\sin\theta$
2. $\cos(-\theta) = \cos\theta$
3. $\tan(-\theta) = -\tan\theta$
4. $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$
5. $\sec(-\theta) = \sec\theta$
6. $\cot(-\theta) = -\cot\theta$.

Trigonometric Identities:

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\sec^2\theta - \tan^2\theta = 1$
3. $\mathrm{cosec}^2\theta - \cot^2\theta = 1$

5.3. ADDITION AND SUBTRACTION FORMULAE:

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\sin(A-B) = \sin A\cos B - \cos A\sin B$.
3. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
4. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

5.4. MULTIPLE ANGLE IDENTITIES:

1. $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$
2. $\cos 2A = \cos^2 A - \sin^2 A$

$$= 1 - 2\sin^2 A$$

$$= 2\cos^2 A - 1$$

$$= \frac{1-\tan^2 A}{1+\tan^2 A}$$

3. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
4. $\sin^2 A = \dfrac{1-\cos 2A}{2}$
5. $\cos^2 A = \dfrac{1+\cos 2A}{2}$

6. $\sin 3A = 3 \sin A - 4 \sin^3 A$

7. $\cos 3A = 4 \cos^3 A - 3 \cos A$

8. $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

Sub-multiple angle formulae:

1. $\sin A = 2 \sin A/2 \cos A/2 = \dfrac{2 \tan A/2}{1 + \tan^2 A/2}$

2. $\cos A = \cos^2 A/2 - \sin^2 A/2$

$$= 1 - 2 \sin^2 A/2$$

$$= 2 \cos^2 A/2 - 1$$

$$= \frac{1 - \tan^2 A/2}{1 + \tan^2 A/2}$$

3. $\tan A = \dfrac{2 \tan A/2}{1 - \tan^2 A/2}$

Half-angle Formulae:

1. $\sin A/2 = \sqrt{\dfrac{1 - \cos A}{2}}$

2. $\cos A/2 = \sqrt{\dfrac{1 + \cos A}{2}}$

3. $\tan A/2 = \sqrt{\dfrac{1 - \cos A}{1 + \cos A}} = \dfrac{\sin A}{1 + \cos A} = \dfrac{1 - \cos A}{\sin A}$

Some more Important Angles:

1. $\sin 18° = \dfrac{\sqrt{5} - 1}{4}$, 2. $\cos 36° = \dfrac{\sqrt{5} + 1}{4}$

5.5. PRODUCT FORMULAE:

1. $\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$
2. $\sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$
3. $\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$
4. $\cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$

Formulae in Terms of tan $A/2$:

Assume $\tan A/2 = t$

1. $\sin A = \frac{2t}{1+t^2}$
2. $\cos A = \frac{1-t^2}{1+t^2}$
3. $\tan A = \frac{2t}{1-t^2}$

5.6. GENERAL SOLUTIONS:

General solutions of $\sin\theta = 0$, $\cos\theta = 0$, $\tan\theta = 0$.

1. If $\sin\theta = 0$ then $\theta = n\pi, n \in z$.
2. If $\cos\theta = 0$ then $\theta = (2n+1)\pi/2, n \in z$
3. If $\tan\theta = 0$ then $\theta = n\pi, n \in z$.

When a trigonometrical equation is solved, among all solutions the solution which is (1) in $[-\pi/2, \pi/2]$ for sine, (2) in $(-\pi/2, \pi/2)$ for tangent and (3) in $[0, \pi]$ for cosine, are the principal values of those functions.

General Solutions of:

1. $\sin\theta = \sin\alpha$
2. $\cos\theta = \cos\alpha$
3. $\tan\theta = \tan\alpha$.

1. If $\sin\theta = \sin\alpha,\ \alpha \in [-\pi/2, \pi/2]$ then
   $\theta = n\pi + (-1)^n \alpha$ where $n \in z$.
2. If $\cos\theta = \cos\alpha,\ \alpha \in [0, \pi]$ then
   $\theta = 2n\pi \pm \alpha$, where $n \in z$.
3. If $\tan\theta = \tan\alpha,\ \alpha \in (-\pi/2, \pi/2)$ then
   $\theta = n\pi + \alpha,\ n \in z$.

5.7. INVERSE TRIGONOMETRICAL FUNCTIONS:

	Function	Domain	Range (Principal value
1.	$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\pi/2 \le y \le \pi/2$
2.	$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
3.	$y = \tan^{-1} x$	R	$-\pi/2 < y < \pi/2$
4.	$y = \text{cosec}^{-1} x$	$x \ge 1$ or $x \le -1$	$-\pi/2 < y < \pi/2$; y
5.	$y = \sec^{-1} x$	$x \ge 1$ or $x \le -1$	$0 < y \le \pi;\ y \ne \pi/2$
6.	$y = \cot^{-1} x$	R	$0 < y < \pi$

5.8. PROPERTIES OF PRINCIPAL INVE TRIGONOMETRIC FUNCTIONS:

1. $\sin^{-1}(\sin x) = x$

   $\cos^{-1}(\cos x) = x$

   $\tan^{-1}(\tan x) = x$

   $\text{cosec}^{-1}(\text{cosec}\, x) = x$

   $\sec^{-1}(\sec x) = x$

   $\cot^{-1}(\cot x) = x$

2. $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}$

$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$

$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$

$\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$

$\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x$

$\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1} x$

3. $\sin^{-1}(-x) = -\sin^{-1} x$

$\cos^{-1}(-x) = \pi - \cos^{-1} x$

$\tan^{-1}(-x) = -\tan^{-1} x$

$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$

$\sec^{-1}(-x) = \pi - \sec^{-1} x$

$\cot^{-1}(-x) = -\cot^{-1} x$

4. $\sin^{-1} x + \cos^{-1} x = \pi/2$

$\tan^{-1} x + \cot^{-1} x = \pi/2$

$\sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2$

5. If $xy < 1$ then

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

6. $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

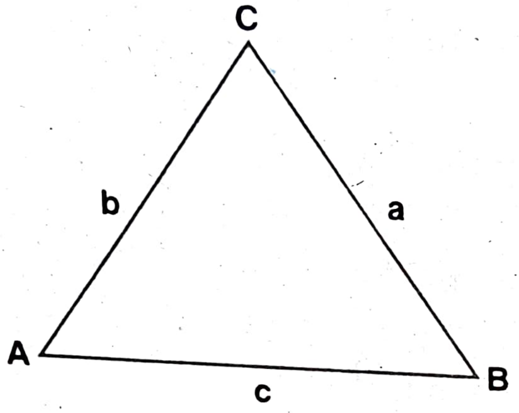
$\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

$$\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$$

$$\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$$

$$\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-xz-yz}\right)$$

5.9. SOLUTIONS TO A TRIANGLE:



1. Sine formula:

In any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the radius of the circumcircle of the triang ABC.

Note: $a = 2R \sin A$; $b = 2R \sin B$; $c = 2R \sin C$

2. Napier's formula:

In any triangle ABC,

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\left(\frac{c}{2}\right)$$

3. Cosine formula:

In any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

Note: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

4. Projection formula:

In any triangle ABC,

$$a = b \cos C + c \cos B$$

5. Sub-multiple angle formulae:

(i) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

(ii) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(iii) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

where $s = \frac{a+b+c}{2}$

6. Tangent formula:

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$$

7. Area formulae: (Δ denotes area of a triangle)

In any triangle ABC,

1. $\Delta = \frac{1}{2} ab \sin C$.

2. $\Delta = \frac{abc}{4R}$

3. $\Delta = 2R^2 \sin A \sin B \sin C$

4. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

5. $\Delta = rs \Rightarrow r = \frac{\Delta}{s}$

6. $\Delta = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin(B+C)}$

7. $\Delta = r^2 \cot A/2 \cot B/2 \cot C/2$

8. $\Delta = S^2 \tan A/2 \tan B/2 \tan C/2$

9. $\Delta = \frac{(s-a)(s-b)(s-c)}{r}$