CHAPTER 13

Differential Calculus

Let y = f(x) be a continuous function. Then, the value of y depends upon the value of x and it changes with a change in the value of x. We use the word increment to denote a small change, i.e., increase or decrease in the values of x and y.

Let Δy be an increment in y corresponding to an increment Δx in x.

Then, $\frac{dy}{dx} = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$. This limit, if it exists finitely, is called the derivative or differential coefficient of y = f(x) with respect to x and is denoted by $\frac{dy}{dx}$ or $f'(x)$ or y_1 . The process of finding the derivative is known as differentiation.

Standard Derivatives

$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}x^n = n.x^{n-1}$	$\frac{d}{dx}x = 1$	$\frac{d}{dx}\frac{1}{x^n} = -\frac{n}{x^{n+1}}$
$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}\frac{d}{dx}x^n = n.x^{n-1}$	$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$	$\frac{d}{dx}\frac{1}{\sqrt{x}} = -\frac{1}{2x\sqrt{x}}$	$\frac{d}{dx}e^x = e^x$
$\frac{d}{dx}e^{mx} = m.e^{mx}$	$\frac{d}{dx}a^x = .a^x.\log_e a$	$\frac{d}{dx}a^{mx} = m.a^{mx}.\log_e a$	$\frac{d}{dx}\log_e x = \frac{1}{x}$

Product and Quotient Rule

$$\frac{d}{dx}u.v = u.\frac{d}{dx}v + v.\frac{d}{dx}u \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

Parametric Functions

Sometimes x and y are given as function of another variable t. Then t is called a parameter. Let x = f(t) and y = g(t), then:

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

Implicit Functions

When the variables x and y are not explicitly or clearly defined in terms of each other ,the function takes an implicit form. We differentiate both sides of the equation term wise, keeping in mind that $\frac{d}{dx}2y = 2.\frac{dy}{dx}$ & $\frac{d}{dt}m^2 = 2m.\frac{dm}{dt}$ and so on.

Function of a Function – Chain Rule

$\text{If } y = f(t) \text{ and } t = g(x), \text{ then } \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$, and the rule can be further extended.

Logarithmic Differentiation – Log Rule

When the given function is a power of some expression or a product of expressions, we take logarithm on both sides and differentiate the implicit functions so obtained.

$\text{If } y = f(x)^{g(x)}, \text{ then; } \log y = g(x).\log f(x)\text{.....Then proceed.}$

Slope – Applied Differentiation

$\text{For } y = f(x), \text{ slope at any point } (x_1, y_1) \text{ is given by } \left.\frac{dy}{dx}\right|_{at\ x_1, y_1}$

Higher Order Derivatives

Let y = f(x) be a differentiable function of x whose second and higher order derivatives exists. The first, second, third, ... and the nth derivatives of this function are denoted by; dy/dx, d^2y/dx^2, d^3y/dx^3, ..., d^ny/dx^n or y_1, y_2, y_3, ... , y_n or f'(x), f''(x), ...

Standard Derivatives

1. If $y = 5x^4 + 3x^3 - 7x^2 + x - 8$. Find dy/dx
   a) $20x^3 + 9x^2 - 14x + 1$
   b) $20x^3 + 9x^2 - 14x$
   c) $20x^3 + 9x^2 + 7x + 1$
   d) None of the above

2. If $y = \sqrt{2x} + 3^{2x}$. Find dy/dx
   a) $2.3^{2x}\log 3$
   b) $\frac{1}{\sqrt{2x}} + 2.3^{2x}\log_e 3$
   c) $\frac{1}{\sqrt{2x}} + 3^{2x}\log_e 3$
   d) None of the above

3. $y = 2x^3 + 3x^2 - 36x + 7$, find the value of "x" for which $\frac{dy}{dx} = 0$.

   a) 2

   b) – 3

   c) 3

   d) Both a) and b) above

4. If $f(x) = x^3 - 2px^2 - 4x + 5$ and $f'(2) = 0$, find "p".

   a) 0

   b) – 1

   c) 1

   d) Both b) and c) above

5. If $y = \log_2 x$. Find dy/dx

   a) $\frac{1}{x \log_{10} 2}$

   b) $\frac{1}{\log_e 2}$

   c) $\frac{1}{x \log_e 2}$

   d) None of the above

6. Find dy/dx for $y = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + ...\infty$.

   a) x

   b) x + y

   c) y

   d) None of the above

7. Find dy/dx, if $y = \dfrac{e^{6\log x} - e^{4\log x}}{e^{2\log x} - \log e}$

a) $2x^3$

b) $4x^3$

c) $-2x^3$

d) None of the above

Product Rule & Quotient Rule

8. $y = 2^x \cdot \log x$

a) $\dfrac{2^x}{x} + \log x$

b) $\dfrac{2^x}{x} + \log x.2^x.\log_e e$

c) $\dfrac{2^x}{x} - \log x.2^x.\log_e 2$

d) $\dfrac{2^x}{x} + \log x.2^x.\log_e 2$

9. $y = \dfrac{x^2-1}{x^2+1}$

a) $2x\left(1+x^2\right)^2$

b) $2x\left(1+x^2\right)^{-2}$

c) $4x\left(1+x^2\right)^{-2}$

d) $4x\left(1+x^{-2}\right)^{-2}$

10. $y = \dfrac{(2x-1)(x+1)}{x-3}$

a) $\dfrac{x^2-6x-1}{(x-3)^2}$

b) $\dfrac{2x^2-12x-2}{(x-3)^2}$

c) $\dfrac{x^2-6x-1}{(x-3)^{-2}}$

d) None of the above

11. y = $\frac{\log x}{2^x}$

a) $2^x\left(x^{-1}-\log x\log 2\right)\left(2^x\right)^2$

b) $2^x\left(x-\log x\log 2\right)\left(2^x\right)^{-2}$

c) $2^x\left(x^{-1}-\log x\log 2\right)\left(2^x\right)^{-2}$

d) None of the above

12. If y = $\frac{x^2}{4x+1}$, find "x" such that f(x) is non-existent.

a) ¼

b) – ¼

c) ½

d) Both a) and b) above

13. If y = $\frac{x^2}{4x+1}$, find "x" such that f'(x) is non-existent.

a) ¼

b) ½

c) – ¼

d) – ½

Parametric Functions

14. Find dy/dx, if $x = t^2$, $y = 2 - t$, at t = 1

a) – ½

b) ½

c) ¼

d) 1

15. Find dy/dx, if x = 3t, y = $\frac{2}{t}$, at t = 2

a) 1/6

b) 1/5

c) – 1/6

d) 1/3

16. Find dy/dx, if $3x = t^3$, $2y = t^2$

a) t

b) t^{-1}

c) t^2

d) 2t

17. Find dy/dx for $x = 6t - t^2$, $y = 2t + 3$, at $t = 0$

    a) ¼

    b) 1

    c) ½

    d) 1/3

18. Find dy/dx, if $x = \frac{1}{1+t}$, $y = \frac{1}{1-t}$

    a) $\frac{-1}{1-t^2}$

    b) $-\frac{1+2t+t^2}{1-2t+t^2}$

    c) Both of the above

    d) None of the above

19. Find the derivative of x^6 wrt x^3.

    a) $2x$

    b) $2x^2$

    c) $2x^3$

    d) $2x^4$

20. Find the derivative of x^6 wrt $\sqrt{x}$.

    a) $12x$

    b) $12x^{5.5}$

    c) $12x^{11}$

    d) $12x^2$

Implicit Functions

Find dy/dx of the following

21. $x^3 + y^3 = a^3$

    a) $\frac{x}{y}$

    b) $-\left[\frac{x}{y}\right]^2$

    c) $\frac{x^2}{y^2}$

    d) None of the above

22. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

a) $-\frac{xb^2}{ya^2}$

b) $\frac{xb^2}{ya^2}$

c) $-\frac{xb}{ya}$

d) None of the above

23. $x^3+3x^2y+y^3=a^3$

a) $\frac{x^2+y^2}{x^2+2xy}$

b) $-\frac{x^2+2xy}{x+y}$

c) $-\left(\frac{x^2+y^2}{x^2+2xy}\right)^{-1}$

d) None of the above

24. $x\sqrt{1+y}+y\sqrt{1+x}=0$

a) $-\frac{1}{(1+x)}$

b) $-\frac{1}{(1+x)^2}$

c) $\frac{1}{(1+x)^2}$

d) None of the above

25. $x^4+x^2y^2+y^4=0$

a) $\frac{x\left(x^2+2y^2\right)}{y\left(2x^2+y^2\right)}$

b) $-\frac{x\left(x^2+2y^2\right)}{y\left(2x^2+y^2\right)}$

c) $-\frac{x\left(2x^2+y^2\right)}{y\left(x^2+2y^2\right)}$

d) None of the above

26. $a^x + a^y = a^{x+y}$

a) a^{x-y}

b) a^{y-x}

c) $-a^{y-x}$

d) None of the above

27. $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\infty}}}$

a) $\frac{1-y^2}{2y-1}$

b) $\frac{1}{2y-1}$

c) $\frac{y^2}{2y-1}$

d) None of the above

28. $\log xy = x^2 - y^2$, at (1, 1)

a) 1

b) ½

c) 1/3

d) ¼

Chain Rule

Find dy/dx of the following functions:

29. $y = \sqrt{x^2 + a^2}$

a) $\frac{-x}{\sqrt{(x^2 + a^2)}}$

b) $\frac{2x}{\sqrt{(x^2 + a^2)}}$

c) $\frac{x}{\sqrt{(x^2 + a^2)}}$

d) $\frac{x}{\sqrt{(x^2 + a^2)^3}}$

30. $y = (3x^2 - 9x + 7)^{-\frac{1}{2}}$

a) $\frac{9-6x}{2\sqrt{(3x^2-9x+7)^3}}$

b) $\frac{9-6x}{\sqrt{(3x^2-9x+7)^3}}$

c) $\frac{-9+6x}{2\sqrt{(3x^2-9x+7)^3}}$

d) $\frac{-9-6x}{2\sqrt{(3x^2-9x+7)}}$

31. $y = \log(x^2 + a)$

a) $\frac{x}{x+a}$

b) $\frac{2x}{x+a}$

c) $\frac{2x}{x^2+a}$

d) $-\frac{2x}{x^2+a}$

32. $y = \log\frac{2x}{1+x^2}$

a) $\frac{1+x^2}{x(1+x)}$

b) $\frac{1-x^2}{x(x^2+1)}$

c) $-\frac{1-x^2}{x(x^2+1)}$

d) $\frac{1+x^2}{x(x^2-1)}$

33. $y = \log\sqrt{1-x^2}$

a) $\frac{x}{(x^2-1)}$

b) $\frac{x}{(1-x^2)}$

c) $\frac{x}{(x^2+1)}$

d) None of the above

34. $y = \log(\log x)$

a) $\frac{x}{\log x}$

b) $\frac{-1}{\log x}$

c) $\frac{1}{\log x}$

d) $\frac{1}{x \log x}$

35. $y = \log\sqrt{x^2 + a^2}$

a) $\frac{1}{x^2 + a^2}$

b) $\frac{x}{x^2 + a^2}$

c) $\frac{-1}{x^2 + a^2}$

d) $\frac{-x}{x^2 + a^2}$

36. $y = \log\log\log\log x$

a) $\frac{1}{\log\log\log x} \cdot \frac{1}{\log\log x} \cdot \frac{1}{\log x} \cdot \frac{1}{x}$

b) $\frac{1}{\log\log x} \cdot \frac{1}{\log x} \cdot \frac{1}{x}$

c) $\frac{1}{\log\log x} \cdot \frac{1}{\log\log x} \cdot \frac{1}{\log x} \cdot \frac{1}{x}$

d) $\frac{1}{\log\log\log x} \cdot \frac{1}{\log\log x} \cdot \frac{1}{\log x}$

37. $y = [\log f(x)]^n$

a) $[\log f(x)]^{n-1} \cdot \frac{1}{f(x)} \cdot f'(x)$

b) $n[\log f(x)] \cdot \frac{1}{f(x)} \cdot f'(x)$

c) $n[\log f(x)]^{n-1} \cdot \frac{1}{f(x)} \cdot f'(x)$

d) None of the above

38. $e^{xy} = 4(1+xy)$, & $e^{xy} \neq 4$

a) $\frac{x}{y}$

b) $\frac{y}{x}$

c) $-\frac{y}{x}$

d) $\frac{y^2}{x}$

Logarithmic Differentiation

39. $y = x^x$

a) $(1+\log x)$

b) x^x

c) $x^x.\log x$

d) $x^x.(1+\log x)$

40. $y = \left(x^x\right)^x$

a) $\left(x^x\right)^x(1+2\log x)$

b) $\left(x^x\right)^x(x+2\log x)$

c) $\left(x^x\right)^x(x+2x\log x)$

d) $\left(x^x\right)^x(x-2x\log x)$

41. $y = x^{x^x}$

a) $x^{x-1} + x^x.\log x(1+\log x)$

b) $x^x.\left[x^{x-1} + x^x.\log x(1+\log x)\right]$

c) $x^{x^x}.\left[x^{x-1} + x^x.\log x(1+\log x)\right]$

d) None of the above

42. $y = x^{\sqrt{x}}$

a) $\frac{x^{\sqrt{x}}}{\sqrt{x}}\left(1+\frac{\log x}{2}\right)$

b) $\frac{x^{\sqrt{x}}}{\sqrt{x}}$

c) $\left(1+\frac{\log x}{2}\right)$

d) None of the above

43. $y = x^{\log x}$

a) $x^{\log x}$

b) $x^{\log x}.2\log x$

c) $\frac{2\log x}{x}$

d) $x^{\log x}\left(\frac{2\log x}{x}\right)$

44. $x^y = e^{x+y}$

a) $\frac{\log x - 2}{(\log x - 1)^2}$

b) $\frac{\log x}{(\log x - 1)}$

c) $\frac{-\log x - 2}{(\log x - 1)}$

d) $\frac{2\log x}{(\log x - 1)^2}$

45. $y.e^y = x$

a) $\frac{x}{y(1+x)}$

b) $\frac{y}{x(1+y)}$

c) $\frac{x}{y(1+y)}$

d) $\frac{y}{x(1+x)}$

46. If $x^m.y^n = (x+y)^{m+n}$

a) $\frac{y}{x}$

b) $-\frac{y}{x}$

c) $\frac{x}{y}$

d) $\frac{y^m}{x^n}$

47. $x^y + y = 1$

a) $\frac{x^{y-1}.y}{x^y.\log x + 1}$

b) $\frac{x^y.y^x}{x^y.\log x + 1}$

c) $-\frac{x^{y-1}.y}{x^y.\log x + 1}$

d) $-\frac{x^{y-1}.y}{x.\log x + 2}$

48. $x^y + xy = 99$

a) $\frac{y + xy}{\log x.x^y + x}$

b) $\frac{y^{y-1} + xy}{\log x.x^y + x}$

c) $-\frac{y + x^{y-1}y}{\log x.x^y + x}$

d) None of the above

49. If $y = \frac{1}{1+x^{n-m}+x^{p-m}} + \frac{1}{1+x^{m-n}+x^{p-n}} + \frac{1}{1+x^{m-p}+x^{n-p}}$, then $\frac{dy}{dx} =$

a) 2

b) 0

c) 1

d) None of the above

50. $y = \log_e \sqrt{x + \sqrt{x^2 + a^2}}$

a) $\frac{1}{\sqrt{x^2 + a^2}}$

b) $\frac{1}{2\sqrt{x^2 + a^2}}$

c) $\frac{2}{\sqrt{x^2 + a^2}}$

d) $\frac{x}{\sqrt{x^2 + a^2}}$

Applications

Find the slope of the following curve at the indicated point:

51. $y^2 = 4ax$, at (1, 2)

a) a

b) – a

c) 4a

d) 2a

52. $\log(xy) = x^2 - y^2 + 2$, at (1, 1)

a) 1

b) –1

c) 1/2

d) 1/3

Higher Order Derivatives

Find $\frac{d^2y}{dx^2}$ in the following cases:

53. $y = x^{10} + 3x^8 + 4x^2 - 7x + 8$

a) $8 + 16x^6 + 9x^8$

b) $90x^8 + 168x^6 + 8$

c) $9x^8 + 16x^6 + 8$

d) $90x^{10} + 168x^8 + 7$

54. $y = \sqrt{x}$

a) $\frac{1}{4x^3}$

b) $\frac{1}{4\sqrt{x}}$

c) $-\frac{1}{4x^{\frac{3}{2}}}$

d) $\frac{1}{4x\sqrt{x}}$

55. $y = \frac{\log x}{x}$

a) $\frac{2\log x - 3}{x^3}$

b) $\frac{\log x - 3}{x^3}$

c) $\frac{\log x - 1}{x^3}$

d) $\frac{2\log x - 3}{x^4}$

56. $x = ct$, $y = \frac{c}{t}$, at t = $\frac{1}{2}$

a) $\frac{8}{c}$

b) $\frac{16}{c}$

c) $\frac{c}{8}$

d) $16c$

57. If $y = \log(x + \sqrt{x^2 + a^2})$, the value of $(a^2 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ would be:

a) 1
b) 3
c) 0
d) –1

58. If $y = \log\left(x + \sqrt{x^2 + 1}\right)$, the value of $\left(1 + x^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ would be:

a) 1
b) 3
c) 0
d) –2